A Network Theory of Project Management*

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Abstract

A principal assigns workers to projects by capping the effort they may exert. Workers differ in which projects they prefer, enjoy collaborating, but find spreading effort across projects costly. Optimal project assignments take a simple nested structure: order projects by the principal's marginal value in the optimum, and worker assignments are derived from this ranking with two idiosyncratic cutoffs. He is unconstrained on the most valuable projects, assigned but capped below what he would freely contribute on a middle interval of projects, and kept off the rest. Effort caps tend to land where a worker most wants to work. Workers sort into specialists and generalists, but standard network centralities do not classify them correctly. The same links that make a worker central make it costly to spread him across multiple project teams. Shocks propagate through the network induced by the project assignment. An improvement local to one project can degrade every other project in the organization.

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1 Introduction

Organizations routinely limit members from doing more of something. The activity is valuable to the organization and the member is willing to contribute more effort, and yet the limit is administered rather than priced outright. Economic theory has largely focused on how organizations shape behavior through incentives, but compensation is not the margin most organizations adjust day to day. Performance pay is unavailable in most public and nonprofit organizations due to legal or budgetary hurdles.[1] Even in corporations with greater contracting freedom, managers have little to no authority over wages. What they do instead is allocate workers to different projects, which has measurable consequences for firm performance as recent empirical work has found (Minni, 2026).[2] Wages are not a substitute in any case: they push effort up at the project they reward, and the problem of interest here is *limiting* it.[3]

As a case in point, consider an associate at a consulting firm who is passionate about a flagship engagement. His staffing manager deliberately holds him below full utilization there and staffs him onto projects for smaller clients he cares less about.[4] How far should the manager hold him back? Pulling him off the flagship frees up capacity for essential work he finds less interesting, but doing so costs the firm effort at the project it cares most about. That cost is not confined to the associate: his colleagues on the flagship project work better with him on it, so holding him back pulls down their contributions as well. This paper studies how managers— through formal limits on worker activities— organize production in the face of these trade-offs by deciding who works on what, how much, and with

[1]Under Title 5 of the United States Code, the yearly monetary incentives a government employee receives cannot exceed a specific threshold. Labor unions frequently file lawsuits against the federal government's implementation of pay-for-performance. See the American Federation of Government Employees' June 2025 memorandum opposing the Office of Personnel Management's overhaul of performance reviews.

[2]See Benson and Shaw (2025) for a broader discussion of the literature on how managers create value through job allocation.

[3]Tools like Jira, Time Doctor and Clockify allow managers to monitor and enforce these limits.

[4]The same assignment-via-cap instrument is present in various other organizations. The NIH Career Development K awards require that recipients devote 75% full-time professional effort to research, capping clinical and other duties at 25%. Engineers in product development have a cap on the number of projects they can join at one time. As multi-project work increases, Wheelwright and Clark (1992) argues that worker productivity may drop because they are not able to allocate their time optimally.

whom.

I develop a model where a principal (she) assigns workers (he) to projects by placing a cap on the effort each worker can exert at each project. I combine the three aforementioned economic forces, each of which has generally been analyzed in isolation and, when considered separately, organizes production differently. First, each worker has an intrinsic preference for each project, which captures fit: his skill set, his taste, the difficulty of the project, etc. (Kristof-Brown et al., 2005; Carpenter and Gong, 2016). Second, he enjoys working alongside others, and effort within a project is a strategic complement (Mas and Moretti, 2009; Chiocchio et al., 2011). Third, effort is costly and costs come from two sources: concentration on a single project and multitasking (Leroy, 2009; Coviello et al., 2014).

The principal values the quality of projects through an increasing and concave function and the quality of a project is given by the sum of its efforts. Formally, she caps workers' efforts, and they play a Nash equilibrium in the induced effort game. I provide conditions for a unique equilibrium and a convex reformulation of the principal's problem in which she directly chooses efforts that satisfy workers' incentive constraints (Proposition 1 and 2).

Effort caps provide the principal with two distinct tools. A positive cap *assigns* a worker to a project, and a zero cap leaves him off of it. Setting a positive cap strictly below what a worker would freely contribute *underutilizes* him. He would like to work more, but is prohibited from doing so, leaving his incentive constraint slack. Setting a cap above what he would like to contribute *fully utilizes* him. Both tools act differently in the organization. Removing a worker from a project eliminates him as a source of collaboration while underutilization keeps him there as one.

Both tools reduce effort and do not directly add it to any project. Holding a worker back at a project costs the principal the effort he would have supplied there *and* the effort his collaborators would have contributed through complementarity. In return, the principal frees up the worker's capacity, which can be spent at other projects. The central tension is that collaboration makes effort caps expensive while multitasking makes them worth it. Collaboration operates within a project and multitasking across them; thus, the principal cannot modify one cap in isolation.

Despite this coupling, optimal project assignments organize project teams in a simple nested structure. Order projects by decreasing marginal value for the principal in the optimum. Each worker's set of assigned and fully utilized projects is sufficiently characterized by two project indices $\gamma_i \leq \sigma_i$ on that ranking. He is fully utilized at the first γ_i projects, assigned but underutilized at every project between γ_i and σ_i, and unassigned at projects beyond σ_i (Theorem 1).

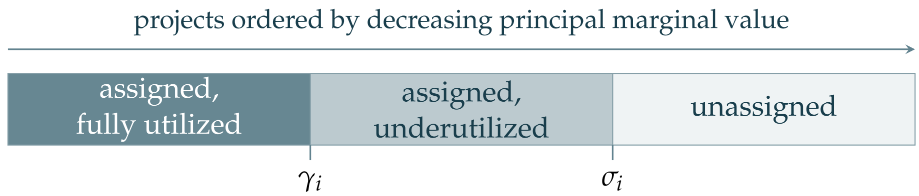


Figure 1: Optimal Worker Project Schedule

That optimal project assignments take an ordered structure is surprising given the heterogeneity in the primitives. One anticipates the principal to exploit workers' comparative advantages and concentrate them on projects they prefer, forming distinct teams with strong complementarity and assignments overlapping irregularly, if at all. Theorem 1 rules this out. Suppose a worker is assigned to a low-value project but underutilized at a high-value project. The principal would like to move his effort from the low-value project to the high-value one. Unfortunately, this is not feasible because he is a source of complementarity for the teammates he leaves behind. Working less on the low-value project jeopardizes other workers' incentive to work there.

The key insight is to coordinate the profitable effort transfer on the *entire team*. Every worker at the low-value project reduces effort by exactly the amount he loses in complementarity, which frees up capacity for them to work more on the high-value project. Workers with strong ties to the underutilized worker see a larger shift in their efforts compared to those with weaker ties. Mathematically, the effort transfer is read off the team's interaction matrix, ensuring every worker's incentive constraints remain satisfied. Thus, employing a worker at a low-value project while underutilizing at a high-value one presents an arbitrage, which the optimal worker project schedule forbids.

When does the principal keep a worker off a project or underutilize him at one? I provide a simple test in the primitives (Proposition 4). Misalignment between effort in the unconstrained equilibrium and her valuation compels the principal to intervene. Effort caps tend to land on projects that inherently attract a lot of effort on their own, and so a project the principal weights heavily in her valuation could be at the bottom of the marginal value ranking. Example 1 illustrates this inversion: the principal and two workers rank three projects in the same fashion, but she underutilizes a worker at two projects he prefers most. This is the gold-plating problem in management where workers over-deliver on a project driven by personal interest, and it is what the consulting associate in the opening paragraph does. Contracts that reward effort cannot reproduce this because the two instruments operate in opposite directions: an effort cap is applied where the principal wants less effort to induce a gain elsewhere, whereas the contract must reward a worker where she wants more effort, which induces a loss elsewhere (Proposition 7).

While Theorem 1 provides structure, it does not predict where the project cutoffs lie because the ranking of projects is endogenous. To give sharper insights, I assume the principal's objective is Cobb-Douglas and vary worker attributes one dimension at a time while holding the others fixed. In each of the following environments, the marginal value ordering of projects in the optimum aligns with their exogenous Cobb-Douglas weights.

When workers have identical preferences, synergies and costs, the assignment margin disappears because every worker works on every project in a worker-symmetric optimal project assignment. A single project index divides the ranking: every worker is fully utilized at projects above it and underutilized at projects below it (Proposition 5). With higher multitasking costs, stronger complementarities, or a larger worker pool, the number of projects with full utilization shrinks. Concentrating effort towards projects at the top is more beneficial, and the principal caps workers on more projects at the bottom to force this imbalance. Higher specialization costs expand the number of fully utilized projects for the opposite reason.

Reintroducing heterogeneity turns the assignment margin back on, and the question of interest is which attributes sort workers. Multitasking costs alone sufficiently sort workers (Theorem 2). Workers with high multitasking costs specialize in only the most valuable projects, while workers with low multitasking costs serve as generalists and are assigned a wide range of projects. The extensive margin is ordered while the intensive margin is not. Whether a specialist works more or less than a generalist on a project they share is sensitive to the parameters.

When workers differ in how well they collaborate, it is natural to believe that highly central workers, because they are valuable to many project teams, should be spread across them. However, standard network game statistics like Katz-Bonacich centrality get the ordering wrong. Designating a central worker to be a generalist diminishes his contributions everywhere due to multitasking costs. The same collaboration links that make a worker valuable precisely make it costly to spread his efforts across them. The correct sorting statistic, which I call a *congestion index*, that captures this trade-off turns out to be a ratio of two Katz-Bonacich centralities with different weightings (Theorem 3). Moreover, the relevant network of interaction shrinks as we descend the project ordering. Once a specialist is unassigned from a project, he is no longer a source of complementarity for any worker left at that project, and so congestion indices are computed recursively on a series of shrinking networks.

Fixing an optimal project assignment, I derive its comparative statics. Through effort caps, the principal chooses which links are active and the magnitude of complementarity generated in equilibrium, and so the comparative statics are those of an endogenous network. I analyze shock propagation across three distinct regimes, ordered by increasing principal involvement

When every worker is assigned to and fully utilized at every project, the incentive constraints pin down every effort and the principal's valuation plays no role. Shocks decompose into a within-project spillover and a network reallocation effect (Proposition 6). For intuition, consider raising worker i's preference for working on project p. Collaboration encourages every worker to work more on project p, but the increase in total efforts raises marginal costs. Workers withdraw from other projects, which is compounded by a loss in complementarity there. A positive shock local to one project degrades the quality of all other projects within the organization. Once some workers are left off some projects, the network reallocation effect operates differently. A worker may reduce effort at a project relative to how much he adds to the project experiencing the positive shock, freeing up his capacity to work more on a third project. The same shock could *increase* the quality of another project, and whether it does depends on the route the shock travels through in the organization (Theorem 4).

With underutilization, the comparative statics behave differently because incentive constraints are no longer sufficient for pinning down optimal efforts. The principal has direct control over efforts at slack incentive constraints, enabling her to substitute one worker's effort for another whenever a shock obliges her to do so. Shocks admit a Slutsky decomposition into an income and substitution effect (Theorem 5). The income effect collects the two channels above, while the substitution effect is determined by the curvature of the principal's valuation.

1.1 Related Literature

This paper contributes to three areas of research: network games, principal-agent theory and organization design.

Network games. Early theories of network games restrict workers to choosing a single action with either complementary or substitutable interactions between neighbors (Ballester et al., 2006; Bramoullé and Kranton, 2007; Bramoullé et al., 2014). Two recent papers, Claveria-Mayol et al. (2024) and Dasaratha et al. (2025) embed a network game within a moral-hazard problem and characterize optimal contracts. My paper differs along three dimensions. First, the principal in both papers above faces a single project; in my model, she divides worker efforts across multiple projects. Second, rather than incentive provision, the principal in my model uses effort caps as her instrument of choice. Third, both papers implicitly take the set of active agents as given. My analysis explicitly characterizes the set of active agents i.e. the assigned agents in a multi-project setting.

Work on network interventions posits a social planner who removes players or designs a vector of marginal benefits to influence activity (Belhaj and Deroïan (2018); Galeotti et al. (2020, 2025)). The principal in my model instead controls workers' action spaces. By determining how much complementarity workers transmit within a project, she effectively selects which links within a network are active, and to what extent. The network is designed, whereas in the network formation literature, link formation is decentralized (e.g. Jackson and Wolinsky (1996); Bala and Goyal (2000)).

This paper directly builds on network games in which players choose multiple actions. In multiplex network games, players act against each other across several layers subject to an aggregate effort constraint (Chen et al. (2018); Zenou and Zhou (2026)).[5] Closest to this work is Sadler and Yeh (2026), in which players select neighbor-specific efforts subject to a convex cost of total effort. Workers in my model choose project-specific efforts which exhibit strategic complementarities and positive spillovers within a project. The effort game is a parametric multigraph version of Sadler and Yeh (2026). The novelty is that the principal selects the structure of the effort game, and the comparative statics pertain to an endogenous network. In a separate project, I study the optimality of modular task assignments where effort at a task provides positive spillovers but are strategic *substitutes* for locally related tasks, which introduces undesirable free-riding in teams (Yeh, 2026).

Principal-agent theory. Models of principal-agent problems have focused on incentive design in multitask or multi-agent settings (Itoh, 1991; Holmstrom and Milgrom, 1991; Dewatripont et al., 2000; Che and Yoo, 2001). This paper combines both dimensions and replaces the contracting instrument with assignments.

Within this literature, assignment is most closely related to delegation. Setting an effort cap is akin to choosing a delegation set (Alonso and Matouschek, 2008), as applied in project choice (Armstrong and Vickers, 2010) and multidimensional actions (Frankel, 2014). I highlight three differences. First, delegation is traditionally motivated by asymmetric information. The principal balances granting an agent flexibility to use his private information while limiting undesirable actions resulting from agent bias. My setting has no private information. Restricting action sets is valuable to the principal because effort is scarce due to convex costs; forbidding effort at one project allows the principal to obtain more effort at another. Second, the nature of actions that are excluded is reversed. In delegation, the principal oftentimes restricts the agent from taking actions the principal dislikes; in my model, the principal precludes a worker from taking actions both enjoy. Third, the delegation sets in my environment are interdependent. Adjusting a worker's effort cap at one project necessarily affects his action set at another project through substitutability and other workers' action sets at the same project through complementarity.

[5] Applications studied include conflict (Franke and Öztürk, 2015; Xu et al., 2022) and communication networks (Demange, 2025)).

Organization design. With transferable utility and output that is supermodular in agents' types, the classic result by Becker (1973) finds the optimal assignment to be positively assortative. In most settings, agents are sorted by a single scalar attribute (e.g. Kremer (1993)). The richness of my environment prevents the principal from sorting agents by any single attribute. A worker who is very good at multitasking could be a poor teammate, and it is not immediately clear which workers strongly complement each other on different projects. Related to this paper is Freund (2025), who show that specialization within a firm generates coworker complementarity and induces a concentration of talent within a small number of firms. In their model, workers belong to a single firm, and so the question of multi-team membership is moot.[6] In my model, a worker may be assigned to several projects at once, and multitasking costs and complementarity drive the nested structure of project teams.

The resulting structure connects to optimal hierarchies, which arise from limited managerial attention (Geanakoplos and Milgrom, 1991), limited decision-making capacity (Hart and Moore, 2005), communication (Visser, 2000; Ambrus et al., 2013) and grouping activities for synergy (Dessein et al., 2010). These are hierarchies of authority or information. In contrast, the nested hierarchy in an optimal project assignment is a hierarchy of membership ordered by project marginal values.

2 Model

There is one principal (she), a set of workers (he) $N = \{1, ..., n\}$ and a set of projects $P = \{1, ..., m\}$.

Workers. Worker i chooses nonnegative effort $e_i^p \in [0, \bar{e}_i^p]$ to play at each project p where $\bar{e}_i^p > 0$. He receives utility given by the sum of per-project benefits less his total cost of effort.

$$U_i(\mathbf{e}_i, \mathbf{e}_{-i}) = \sum_{p=1}^{m} \underbrace{(\alpha_i^p e_i^p + \sum_{j \neq i} g_{ij} e_i^p e_j^p)}_{\text{per-project benefit}} - \underbrace{\left(\sum_{p=1}^{m} \frac{c_i}{2} (e_i^p)^2 + r_i \sum_{1 \leq p < q \leq m} e_i^p e_i^q \right)}_{\text{cost of effort}}$$

[6] For additional work on the optimal composition of a team and how it is affected by size, diversity, and incentives, see Prat (2002); Georgiadis (2015); Fu et al. (2016); Glover and Kim (2021).

where $\mathbf{e}_i = \{e_i^p\}_{p=1}^m$ and $\mathbf{e}_{-i}$ represents all other workers' efforts at all projects.

At each project p, worker i receives a benefit that is linear in his effort. His intrinsic preference for working on project p is captured by $\alpha_i^p > 0$. The level of effort complementarity between worker i and j is $g_{ij} > 0$, which I assume to be symmetric i.e. $g_{ij} = g_{ji}$.[7]

Worker i incurs a general quadratic cost of effort. His cost of specialization on a single project is $c_i > 0$, while his cost of multitasking is $r_i > 0$. To maintain convexity in costs, I assume that $c_i > r_i$.[8] Because costs are convex and benefits are linear in a worker's own efforts, workers' utilities are concave in their own efforts.

Principal. The quality of project p is given by the sum of worker efforts $\mathcal{E}^p := \sum_{i=1}^n e_i^p$. The principal values project qualities according to $V : \mathbb{R}^m \to \mathbb{R}$, which is strictly increasing in each argument, strictly concave and C^2. The principal's choice of upper bounds $\bar{\mathbf{e}} \in \mathbb{R}_+^{mn}$ solves:

$$\max_{\bar{\mathbf{e}} \geq \mathbf{0}} \; V(\mathcal{E}^1, ..., \mathcal{E}^m)$$

$$\text{s.t. } \mathbf{e} \text{ is the (principal-preferred) pure-strategy Nash equilibrium given } \bar{\mathbf{e}}$$

In general, if there are multiple equilibria induced by $\bar{\mathbf{e}}$, then the one that achieves the highest principal valuation is selected. Subsequently, I provide conditions under which there is always a unique equilibrium so no selection is needed.

2.1 Implementability

In this section, I characterize the feasible set of the principal's problem. Symmetric effort complementarity permits us to write the workers' effort game as an exact potential game. The exact potential is

$$\Phi(\mathbf{e}) = \sum_{i=1}^n U_i(\mathbf{e}) - \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} \sum_{p=1}^m g_{ij} e_i^p e_j^p.$$

[7] The complementarity terms $g_{ij} e_i^p e_j^p$ can be equivalently interpreted as "helping" efforts by colleagues that reduce a worker's cost of effort instead of raising his benefit from working.

[8] The Hessian matrix of worker i's cost of effort is $(c_i - r_i)\mathbf{I}_m + r_i \mathbf{1}_m \mathbf{1}_m^\intercal$, which has eigenvalues $c_i + (m-1)r_i$ and $c_i - r_i$ with multiplicity of $m-1$. For the cost to be strictly convex, its Hessian needs to be positive definite, which is true if $c_i - r_i > 0$.

Assumption 1. *The system interaction matrix, given by*

$$\mathbf{H} := (\mathbf{C} \otimes \mathbf{I}_m + \mathbf{R} \otimes (\mathbf{1}_m \mathbf{1}_m^\intercal - \mathbf{I}_m)) - (\mathbf{G} \otimes \mathbf{I}_m)$$

is positive definite, where $\mathbf{C} := \mathrm{diag}(c_i)_{i \in N}$, $\mathbf{R} := \mathrm{diag}(r_i)_{i \in N}$, $\mathbf{G}$ *has entries* $\mathbf{G}_{ij} = g_{ij}$ *and 0 on its diagonal, and* $\otimes$ *is the Kronecker product.*[9]

Under this assumption, the Hessian of the exact potential is $-\mathbf{H}$, and so the exact potential is strictly concave. It admits a unique maximizer corresponding to the unique Nash equilibrium of the workers' effort game.[10]

The $\mathbf{H}$ matrix has dimension $nm \times nm$ with worker-project pairs for indices. Each element captures an interaction between two worker-project pairs. Working on project q raises worker i's marginal cost at project p by his multitasking parameter r_i, recorded in $\mathbf{H}[(i,p),(i,q)] = r_i$. The same goes for specialization costs. When worker j increases his effort on project p, it raises worker i's marginal benefit at project p through collaboration, represented by $\mathbf{H}[(i,p),(j,p)] = -g_{ij}$. There are no explicit interactions between worker i at project p and worker j at project q i.e. $\mathbf{H}[(i,p),(j,q)] = 0$ because synergies between workers operate within a project, and multitasking costs affect a single worker across multiple projects. Nevertheless, interactions between different workers at different projects surface in equilibrium.

Uniqueness of the Nash equilibrium guarantees that no selection rule is required in the principal's problem. The following result ensures it is without loss for the principal to directly choose efforts so long as each worker's *incentive constraints* are respected.

Proposition 1. *For* $\mathbf{e} \geq \mathbf{0}$, *there exists an upper bound* $\bar{\mathbf{e}}$ *that implements* $\mathbf{e}$ *as an equilib-*

[9] Formally, given a $m \times n$ matrix $\mathbf{A}$ and a $p \times q$ matrix $\mathbf{B}$, the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is a $mp \times nq$ block matrix given by $\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$.

[10] There are other ways of showing a Nash equilibrium exists and is unique. One is to write down each worker's best response function and then appeal to Brouwer's Fixed Point Theorem. Uniqueness follows from applying the concave games result in Rosen (1965). The fixed-point iteration does not provide a practical approach to computing the Nash equilibrium. Another is to cast the equilibrium as the optimum of a constrained quadratic program, where the quadratic form is the exact potential.

rium if and only if

$$\alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p \geq c_i e_i^p + r_i \sum_{q \neq p} e_i^q$$

whenever $e_i^p > 0$*. Moreover, the principal can implement* $\mathbf{e}$ *by setting* $\bar{\mathbf{e}} = \mathbf{e}$*.*

Proof. See Appendix A.1. □

When a worker's marginal utility from working on a project is nonnegative, reducing effort is never profitable. Setting the effort cap to be exactly the desired effort level prevents the worker from contributing more even when his marginal utility is strictly positive. I call the condition outlined in Proposition 1 worker i's **incentive constraint** at project p. Due to the linear-quadratic framework, each incentive constraint is linear in worker efforts and defines a closed half-space.

The left term is worker i's *effective* marginal benefit at project p. It depends on his intrinsic preference and the efforts of other workers at that project. An increase in other workers' efforts at project p relaxes his incentive constraint. The principal can encourage or discourage effort from worker i at project p by raising or lowering his effort cap. The right term is worker i's *effective* marginal cost at project p. To encourage him to work more on project p, raising his effort cap is not enough if his incentive constraint binds. The principal has to reduce his efforts at other projects by lowering effort caps, or by encouraging more effort from other workers at that project.

2.2 Managerial Tools: Assignment and Utilization

By controlling effort caps, the principal not only directs efforts towards projects she cares about, but also shapes the pattern of interaction among workers.

Definition 1. Given effort profile $\mathbf{e}$, worker i is **assigned** to project p if $e_i^p > 0$, otherwise he is **unassigned**. Worker i is **fully utilized** at project p if his incentive constraint binds i.e.

$$\alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p = c_i e_i^p + r_i \sum_{q \neq p} e_i^q.$$

He is **underutilized** if

$$\alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p > c_i e_i^p + r_i \sum_{q \neq p} e_i^q.$$

Keeping worker i off project p nullifies (i, p) as a source of any interactions. That is, worker i at project p generates no complementarity for others at p or substitutability for his effort at other projects. Through underutilization, the principal deliberately keeps worker i as a participating team member at project p, but keeps his incentive constraint slack there. The effort cap governs his equilibrium effort. Utilization also informs the principal how much latitude she has when setting effort caps. Different effort caps may implement the same effort profile as their unique equilibrium. If worker i is fully utilized at project p for $\mathbf{e}$, then any effort cap $\bar{e}_i^p \geq e_i^p$ is sufficient. In effect, the principal can leave him *unconstrained* at project p. However, for a worker i who is underutilized at project p, the principal must set a binding cap $\bar{e}_i^p = e_i^p$. Setting a higher effort cap induces worker i to increase his equilibrium effort at project p. A different equilibrium results because i necessarily shirks at other projects in order to work more on p.

2.3 A Convex Optimization Problem

A zero effort cap forces a worker to play zero effort regardless of his marginal utility. This complicates the principal's problem because incentive constraints are effectively ignored at zero effort even when marginal utilities are negative. Which incentive constraints are active depends on whether effort is positive or zero, so tracking them requires a separate binary variable, resulting in a nonconvex feasible set. In Appendix B, I write the principal's problem as a mixed-integer nonlinear program and give an explicit example of this nonconvexity.

There are two reasons why a worker exerts zero effort at a project in equilibrium: (1) the principal did not assign him even though he was willing to work there, or (2) his marginal utility at that project evaluated at zero effort is negative. The second case is special because no cap has bite. His effort is zero no matter whether the principal caps him at zero or leaves him unconstrained. The scenarios described in the introduction (a split on a physician's time, cap on project participation, utilization target for a consultant) are of the first kind, and it is the exclusion of this type that I study as the managerial instrument. I restrict attention to parameters under which a worker is always weakly motivated to start work on a project, but is forced off a project only if the principal has set his effort cap there to zero.

The incentive constraint for worker-project pair (i, p) with zero effort is

$$r_i \sum_{q \neq p} e_i^q \leq \alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p.$$

To ensure worker i does not abandon project p before starting it, the multitasking costs cannot be too large relative to his intrinsic preference and his total effort. To provide a bound on his total effort in an effort profile that respects every incentive constraint, I introduce the *aggregate interaction matrix.*

Definition 2. The **aggregate interaction matrix** is $\mathbf{P} := \mathbf{C} + (m-1)\mathbf{R} - \mathbf{G}$.

The following lemma demonstrates how $\mathbf{P}$ earns its name. For notation, let $\mathbf{E} := [\sum_{p=1}^m e_1^p \cdots \sum_{p=1}^m e_n^p]$ be the vector of workers' total efforts.

Lemma 1. *For any effort profile where every incentive constraint is satisfied,* $\mathbf{E} \leq \tilde{\mathbf{E}} := \mathbf{P}^{-1} \sum_{p=1}^m \boldsymbol{\alpha}^p$.

Proof. See Section A.2. □

The inverse of the $\mathbf{P}$ matrix provides an upper bound $\tilde{\mathbf{E}}$ on the aggregate effort each worker exerts in any effort profile respecting every incentive constraint. The proof rests on the fact that $\mathbf{P}^{-1}$ is nonnegative, which is a direct consequence of Assumption 1.[11] It is not yet a bound on what the principal can *implement* because we have not ruled out workers violating their incentive constraints at zero effort. The following condition does so.

Assumption 2. *For each worker i, $r_i \tilde{E}_i \leq \min_p \alpha_i^p$ where $\tilde{\mathbf{E}} := \mathbf{P}^{-1} \sum_{p=1}^m \boldsymbol{\alpha}^p$.*

The condition limits how unevenly a worker's preferences can be distributed. A worker's multitasking burden never outweighs his preference for his weakest

[11]The entries of $\mathbf{P}^{-1}$ contain rich economic content: they capture how a worker's total effort responds to changes in total preferences for other workers. This observation will be leveraged in Section 5 regarding comparative statics.

project, guaranteeing he is always weakly motivated to start work anywhere. Scaling a worker's preferences across all projects equally does not affect the inequality. When a worker prefers each of his projects the same, a sufficient condition in the primitives for the assumption is $c_i - r_i \geq r_i(\sum_{j \neq i} g_{ij}/r_j)$. Recall that the model already assumes that $c_i - r_i > 0$. Assumption 2 asks for more: the curvature needs to be large enough to dominate the pull from his collaborators, the latter of which decreases the worse his collaborators are at multitasking. Most importantly, it does not restrict how workers are ranked against one another, such as who multitasks well, who collaborates better with whom, and which projects workers like least. These are the comparisons that power the main structural results.

The following result formally shows that under this condition, the set of implementable effort profiles is equivalent to the set of effort profiles that satisfy incentive constraints for all worker-project pairs. Ultimately, this allows us to formulate the principal's problem as a tractable convex optimization problem.

Proposition 2. *Under Assumption 2, the total effort vector* $\mathbf{E}$ *for any implementable effort profile satisfies* $\mathbf{E} \leq \tilde{\mathbf{E}}$*. The set of implementable effort profiles is equivalent to the set of effort profiles that satisfy incentive constraints for all worker-project pairs and the principal's problem is:*

$$
\begin{aligned}
&\max_{\mathbf{e}} \; V(\mathcal{E}^1, ..., \mathcal{E}^m) \\
&s.t. \;\; \mathbf{e} \geq \mathbf{0}, \\
&\forall (i, p), \; \alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p \geq c_i e_i^p + r_i \sum_{q \neq p} e_i^q \\
&\forall p, \; \sum_{i=1}^{n} e_i^p = \mathcal{E}^p
\end{aligned}
$$

Proof. See Section A.3. □

A solution to the principal's problem exists because the objective is continuous and the feasible set is closed and bounded. Moreover, the solution is unique in total efforts because the objective is strictly concave over total efforts and the feasible set is convex.

2.4 Remarks

1. There are no explicit contracts in this model. Worker i's intrinsic preference for project p, α_i^p, reflects his personal motivation, taste, skill set, talent, project difficulty, etc. In Section 6.1, I consider an alternate model where the principal rewards effort. To achieve underutilization without an effort cap, the principal needs to implement *negative* wages.

2. Symmetry in complementarity is sufficient for a unique equilibrium. Asymmetric complementarities, discussed in Section 6.2, do not affect the main structural results.

3. In the model, there is no upper limit on the efforts the workers exert. Imposing an external hard limit on efforts (for example, a 40-hour work week) does not change the main structural results as shown in Section 6.3.

3 Structure of Optimal Project Assignments

The main structural result requires two conditions of the optimum, which I show to be generic in Appendix C.

Assumption 3. *In the optimum, (1) every binding incentive and nonnegativity constraint has strictly positive multipliers and (2) worker i's incentive constraint at project p is slack whenever $e_i^p = 0$.*

Condition (1) assumes strict complementarity at the optimum. For condition (2) to fail, there must be a worker-project pair (i, p) where the incentive constraint exactly binds when $e_i^p = 0$, which is a knife-edge case.

How are projects assigned? At which projects does the principal deliberately hold workers back? The following result shows that assignment and full utilization patterns are ordered and adhere to a common ranking. For notation, let $V^p := \partial V / \partial \mathcal{E}^p$.

Definition 3. The **assignment** and **full utilization sets** are $A^p := \{i \mid e_i^p > 0\}$ and $\mathcal{F}^p := \{i \mid i \text{ is fully utilized at } p\}$.

Theorem 1. *Rank projects by the principal's marginal value in the optimum so that $V^1 \geq \cdots \geq V^m$. There exists natural numbers $0 \leq \gamma_i \leq \sigma_i \leq m$ for each worker i such that worker i is*

- *fully utilized at projects $1, ..., \gamma_i$,*
- *assigned but underutilized at projects $\gamma_i + 1, ..., \sigma_i$,*
- *and not assigned to projects $\sigma_i + 1, ..., m$.*

This implies

$$A^1 \supseteq \cdots \supseteq A^m \quad \textit{and} \quad \mathcal{F}^1 \supseteq \cdots \supseteq \mathcal{F}^m.$$

Proof. See Appendix A.4. □

Optimal project assignments are simple. For each worker, two cutoffs determine which project he is assigned to and fully utilized at. Importantly, his assignment schedule comprises three successive regimes. For the most valuable projects on the margin, the principal sets relatively high effort caps so that he is working as much as he is willing to. For medium-value projects, the worker is assigned but capped. He would like to contribute more, but the principal prohibits him from doing so. On the remaining low-value projects, he is unassigned. All worker schedules agree on the *same* ranking of projects, namely by the principal's marginal value in the optimal. Project teams are necessarily nested in the optimum.

That optimal project assignments embody such a clean structure is surprising, especially because all parameters— preferences, synergies, and costs— are heterogeneous. We expect workers to retain comparative advantage at certain projects over others, and it seems natural that workers specialize in distinct projects and have non-overlapping project assignments. Theorem 1 precludes any worker from being capped or excluded from a project while being fully utilized or assigned to a less valuable one.

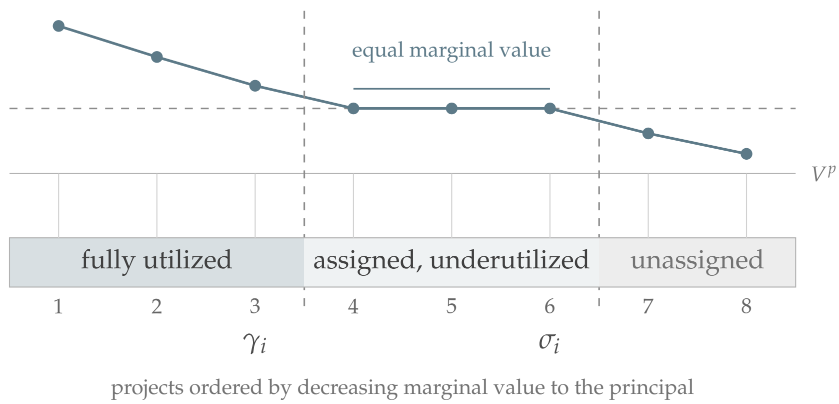


Figure 2: Optimal Worker Project Schedule

Why then should assignments nest in the optimum? There are two halves to the argument. Between projects the principal values differently on the margin, nesting follows from a transfer of efforts that would otherwise be profitable (Section 3.1 and Section 3.2). Suppose a worker is assigned to a low-value project while holding slack at a high-value one. Moving his efforts alone is infeasible as it reduces complementarity from the project team he leaves behind. Instead, I design a *team-coordinated* effort transfer from the low-value project to the high-value one that is both feasible and profitable. Between projects with the same valuation, such a transfer is unavailable. The proof is more technical and leverages a linear complementarity argument (Section 3.3). It turns out that these are exactly the projects at which a worker is assigned but underutilized, as depicted in the figure above.

3.1 The Within-Project Interaction Matrix

Before designing a successful effort transfer, we first need a method of tracking how the slack of an incentive constraint changes when efforts are moved from one project to another:

$$s_i^p := \alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p - c_i e_i^p - r_i \sum_{q \neq p} e_i^q$$

Worker i is fully utilized at project p when $s_i^p = 0$, and underutilized when $s_i^p > 0$. His total effort is sufficient for determining his marginal cost at project p when we substitute total effort $E_i := \sum_{q=1}^m e_i^q$ and rewrite $\sum_{q \neq p} e_i^q = E_i - e_i^p$. Stack the slack equations for all workers at project p to obtain:

$$\mathbf{s}^p = \boldsymbol{\alpha}^p - (\mathbf{C} - \mathbf{R} - \mathbf{G})\mathbf{e}^p - \mathbf{R}\mathbf{E}$$

The following matrix is vital to understanding how efforts and slacks move together.

Definition 4. The **within-project interaction matrix** is $\mathbf{M} := \mathbf{C} - \mathbf{R} - \mathbf{G}$.

Efforts in the entire system reach a worker's incentive constraint at project p through two objects: the vector of project efforts $\mathbf{e}^p$ and the vector of total efforts $\mathbf{E}$. The $\mathbf{M}$ matrix is the operator on the first, while everything a worker does at other projects enters through $\mathbf{RE}$ and nowhere else.

The off-diagonals of $\mathbf{M}$, which are $-g_{ij}$, capture complementarity between workers. The diagonal term $c_i - r_i$ measures worker i's marginal cost of effort at project p, holding his total effort fixed. An extra unit of effort at project p costs c_i, and to keep total effort constant, he must give up a unit of effort somewhere else for a refund of r_i. In other words, $c_i - r_i$ is the cost of reallocation towards project p for worker i.[12]

3.2 A Team-Coordinated Transfer

Consider a effort transfer vector $\mathbf{x} \geq \mathbf{0}$ from project q to project p where x_j is the transfer amount for worker j i.e. $\Delta\mathbf{e}^p = \mathbf{x}$, $\Delta e^q = -\mathbf{x}$ and $\Delta\mathbf{e}^t = \mathbf{0}$ for all $t \neq p, q$. While multitasking costs respond to how a worker distributes his effort, they only reach his incentive constraints at other projects through his total effort. Because total efforts are kept constant through $\mathbf{x}$, this transfer is local to p and q.

For a transfer to be feasible, it must respect incentive constraints at both projects. The $\mathbf{M}$ matrix exactly captures how slacks at projects p and q move: $\Delta\mathbf{s}^p = -\mathbf{M}\mathbf{x}$, $\Delta\mathbf{s}^q = \mathbf{M}\mathbf{x}$ and $\Delta\mathbf{s}^t = \mathbf{0}$ for all $t \neq p, q$. Worker j's slack at p moves for three reasons: (1) his extra effort at p costs $c_j x_j$, (2) reducing effort at q refunds $r_j x_j$ and (3)

[12] Alternatively, the Hessian of worker i's cost is $(c_i - r_i)\mathbf{I}_m + r_i \mathbf{1}_m \mathbf{1}_m^\intercal$, and so reallocations that keep total effort fixed are exactly the directions orthogonal to $\mathbf{1}_m$. It is an $(m-1)$-dimensional eigenspace the Hessian applies $(c_i - r_i)$ to.

his collaborators who raised their efforts at project p supply $\sum_{k \neq j} g_{jk} x_k$. His slack at project q moves in the opposite direction.

Now, assume there are two projects p and q where $V^p > V^q$ and worker i is assigned to q but has slack in his incentive constraint at p. A naïve transfer only moves worker i's effort from q to p. Formally, the principal sets $\mathbf{x} = \delta \boldsymbol{\epsilon}_i$ where $\delta > 0$ and ϵ_i is the i-th standard basis vector. This is profitable for small enough δ because

$$\Delta V = (V^p - V^q)\delta + O(\delta^2)$$

What is wrong with this transfer? At project p, worker i's slack falls and every other worker's incentive constraint relaxes from the extra complementarity. The problem is at project q. A loss in complementarity from worker i makes other workers less willing to work at q, potentially leading to violations in their incentive constraints.

The key insight is to coordinate the transfer on the *entire project team* rather than just worker i. Because the principal wants worker i to exert more effort on project p, this presents an opportunity to shift other workers' efforts from q towards p as well. The interactions captured within $\mathbf{M}^{(A^q)}$, the principal submatrix of $\mathbf{M}$ restricted to workers assigned to project q, pin down exactly how to accomplish this. Consider the following transfer:

$$x_j = (\delta(\mathbf{M}^{(A^q)})^{-1}\boldsymbol{\epsilon}_i)_j \text{ for } j \in A^q \text{ and } x_j = 0 \text{ for } j \notin A^q$$

The inverse matrix $(\mathbf{M}^{(A^q)})^{-1}$ exists and is nonnegative, implying the transfer has at least one positive entry.[13] The principal transfers more effort among workers who are strong collaborators with i than those who are weak collaborators. Such a transfer is profitable for sufficiently small δ because

$$\Delta V = (V^p - V^q)\mathbf{1}_n^\intercal \mathbf{x} + O(||\mathbf{x}||^2)$$

By customizing the transfer to $(\mathbf{M}^{(A^q)})^{-1}$, the principal ensures all incentive constraints remain satisfied. I demonstrate this is true at project q, which is where

[13] Formally, $\mathbf{M}^{(A^q)}$ is a M-matrix because it inherits positive definiteness from $\mathbf{M}$ by Sylvester's criterion and has negative off-diagonals. $\mathbf{M}$ is itself positive definite by Assumption 1.

the naïve transfer failed. Let $\overline{A^q}$ be the set of all workers not assigned to project q. Partition $\mathbf{M}$ into blocks according to whether a worker is assigned to project q. The following computation is instructive.

$$\Delta \mathbf{s}^q = \mathbf{M}\mathbf{x} = \begin{bmatrix} \mathbf{M}^{(A^q)} & \mathbf{M}^{(A^q),(\overline{A^q})} \\ \mathbf{M}^{(\overline{A^q}),(A^q)} & \mathbf{M}^{(\overline{A^q})} \end{bmatrix} \begin{bmatrix} (x_j)_{j \in A^q} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{(A^q)} (x_j)_{j \in A^q} \\ \mathbf{M}^{(\overline{A^q}),(A^q)} (x_j)_{j \in A^q} \end{bmatrix}$$

The first $|A^q|$ rows are the incentive constraints of the project team at q, and this is where the transfer $\mathbf{x}$ operates. Each worker cuts back at q by exactly the amount that offsets the complementarity he loses when his teammates cut back alongside him. Mathematically, $\mathbf{M}^{(A^q)}(x_j)_{j \in A^q} = \delta \boldsymbol{\epsilon}_i$, and so worker j's slack at q remains the same as before the transfer. Only worker i's slack at q changes; it rises by δ. The remaining rows correspond to workers not on the team at project q. They lose a bit of slack, but Assumption 3 gives them strictly positive slack to absorb the loss. All of the same observations hold in reverse for project p.

Stated plainly, assigning a worker to a project with low marginal value but holding him back on one with high marginal value presents an arbitrage. The strategy is to redistribute the entire project team's effort towards the high-value project while respecting team members' incentive constraints. The mathematical object of interest is the project team-specific interaction matrix $\mathbf{M}^{A^q}$, which factors in how workers adjust their efforts in the transfer.

3.3 Projects with Equivalent Marginal Values

The transfer argument is silent for projects that share the same marginal value because there is no profitable deviation to exhibit. The ordering of worker project schedules in Theorem 1 requires tied projects to obey nested staffing patterns. The next result is stronger— these projects necessarily have the same assignment and full utilization sets.

Proposition 3. *Projects with equal marginal value in the optimum have identical assignment and full utilization sets. Moreover, each worker's underutilized projects share a common marginal value* $V^{\gamma_i+1} = \cdots = V^{\sigma_i}$.

Proof. See Section A.5. □

Team composition could in principle be a matter of coordination i.e. worker i works on p because j is there, and j works on p because i is there. Two equally valuable projects could be staffed with different teams. Assumption 1 rules this out. It ensures the within-project interaction matrix $\mathbf{M}$ is positive definite, keeping complementarity within a project below what it costs a worker to reallocate effort. Projects with the same marginal value pose the same staffing problem for the principal, and because there is no coordination issue, they must have identical assignment and full utilization sets. I formally establish this uniqueness with linear complementarity techniques in the appendix.

The second statement can be directly shown using a transfer argument. Suppose the principal underutilizes him at two projects, p and q, in the optimum. She can perform the same team-coordinated effort transfer as before if there is an imbalance in marginal values. For example, if $V^p > V^q$, then she can pull efforts from q towards p, and vice versa, for a profitable deviation. Therefore, it must be that both projects share the same marginal value.

It is tempting to read Proposition 3 as making the underutilization interval fragile, as their marginal values must agree. This is false precisely because the effort for an underutilized worker-project pair is defined by two *strict* inequalities on the primitives— a slack incentive constraint and a slack nonnegativity constraint. In the following section, Example 1 features underutilization and has a unique optimum. Hence, the underutilization pattern there survives small perturbations of every parameter.

3.4 Partial Assignment and Underutilization

When does the principal leave workers off projects or deliberately hold them back? Theorem 1 gives structure to worker project schedules, but does not address when the principal intervenes. The following is an explicit example of an optimal project assignment in a two-worker, two-project setting. Interestingly, the principal underutilizes a worker at projects that both the worker *and* the principal prefer most given the model primitives.

Example 1. There are two workers i and j and three projects p, q and s. Every party ranks the projects the same way. However, the optimum does not.

	Project		
	p	q	s
Preference $\alpha_i^{(\cdot)}$, worker i	0.45	0.35	0.25
Preference $\alpha_j^{(\cdot)}$, worker j	0.60	0.45	0.20
Principal's weight $\beta^{(\cdot)}$	0.75	0.50	0.40

Costs: $c_i = 0.6$, $c_j = 0.7$, $r_i = 0.2$, $r_j = 0.15$.

Synergy: $g_{ij} = g_{ji} = 0.05$.

Valuation: $V = \sum_p \beta^{(p)} \ln \mathcal{E}^p$.

Table 1: Model Parameters

	Project		
	p	q	s
$\hat{\mathbf{e}} = \mathbf{H}^{-1}\boldsymbol{\alpha}$			
Worker i	0.667	0.379	0.069
Worker j	0.795	0.497	0.014
Quality $\mathcal{E}^{(\cdot)}$	1.462	0.876	0.083
Optimal $\mathbf{e}^*$			
Worker i	0.396†	0.289†	0.191
Worker j	0.774	0.492	0.028
Quality $\mathcal{E}^{(\cdot)}$	1.170	0.780	0.219
Marginal Utility			
Worker i	0.155	0.084	0
Worker j	0	0	0
Marginal Value $V^{(\cdot)}$	0.641	0.641	**1.827**
Rank by $V^{(\cdot)}$	2	2	**1**

† Underutilized: the incentive constraint is slack.

Table 2

With full utilization, both workers exert a lot of effort at project p and ignore s. The principal does the opposite. She suppresses both workers' efforts at project p so that they can work more on project s. The principal optimally underutilizes worker i at project p and q, which are the projects that both the principal *and* worker i prefer most. The underutilization instrument bites exactly where she values effort least on the margin. Project s carries the smallest weight but the largest marginal value precisely because both workers work so little at it if the principal does not set any caps. Assignments are vacuously nested because $A^s = A^q = A^p$, while full utilization sets are strictly nested because $\mathcal{F}^s = \mathcal{F}^q \supset \mathcal{F}^p$.

The example showed that even if both workers agreed with the principal on their preferences over the projects, the principal still held a worker back on projects he most preferred. There is a level of misalignment between the principal's valuation and the combination of worker attributes in the unconstrained equilibrium, which the following result formalizes.

Proposition 4. *The unconstrained equilibrium $\hat{\mathbf{e}}$ is optimal if and only if $\nabla_{\mathbf{e}} V(\hat{\mathbf{e}})$ lies in the convex cone generated by the columns of $\mathbf{H}$.*

Proof. See Appendix A.6. □

Through a combination of (partial) assignment and (under)utilization, the principal adjusts workers' efforts so that they align her valuation on the margin, represented by the gradient of V, with the columns of $\mathbf{H}$. This result provides a simple diagnostic of when the principal should underutilize workers or leave them off projects entirely. This computation uses only the primitives of the model. If $(\mathbf{H})^{-1}\nabla_{\mathbf{e}} V(\hat{\mathbf{e}}) \geq 0$, then no adjustment of effort caps is required; simply let the workers choose efforts as they desire.[14] On the other hand, if $(\mathbf{H})^{-1}\nabla_{\mathbf{e}}(\hat{\mathbf{e}})V$ yields a negative entry, then the principal must intervene in the optimum.

4 Designating Specialists and Generalists

Theorem 1 reduces the structure of optimal project assignments to a single ranking by the principal's marginal value and two cutoffs for each worker: one that represents the boundary between assignment and exclusion, and the other between full utilization and underutilization. It is not apparent how these cutoffs are

[14] KKT conditions are both necessary and sufficient in the convex optimization problem.

determined because the ordering of project marginal values is endogenous. In this section, I adopt a parametric approach and vary synergies and costs to see how assignments sort workers into specialists and generalists. The principal values projects according to a log-linearized Cobb-Douglas function.

$$V(\mathcal{E}^1, ..., \mathcal{E}^m) = \sum_{p=1}^{m} \beta^{(p)} \ln(\mathcal{E}^p)$$

Each $\beta^{(p)}$ weight represents project p's relative importance for the principal. Without loss, I order projects by decreasing $\beta^{(p)}$ weight and normalize $\sum_{p=1}^{m} \beta^{(p)} = 1$. The Cobb-Douglas objective satisfies the Inada conditions for each project, and so the principal must assign at least one worker to each project. Before introducing any heterogeneity, I highlight properties of optimal project assignments when workers are identical. For each worker i and project p, the parameters are $\alpha_i^p \equiv \alpha$, $r_i \equiv r$, $g \equiv g_{ji}$.

Proposition 5. *There exists a worker-symmetric optimal project assignment with the following properties.*

1. *Every worker is assigned to every project.*
2. *There exists a single project index γ where every worker is fully utilized at projects $s \leq \gamma$ and underutilized at projects $t > \gamma$.*
3. *The project index γ is non-increasing in r, g and n, non-decreasing in c and independent of α.*

Proof. See Appendix A.7. □

The assignment margin shuts down in the worker-symmetric optimum, allowing us to isolate and focus on utilization, which is characterized by a single project index γ. All else equal, raising multi-tasking costs r or collaborative gains g makes it more valuable for the principal to concentrate more effort on a smaller set of projects. When workers struggle to work on many projects simultaneously, the principal limits their utilization on more projects, pushing them to exert higher effort on more valuable projects. The principal capitalizes on higher gains from collaboration to extract higher quality at more valuable projects at the expense of the less valuable ones. An increase in team size n likewise strengthens synergies, and so it operates in the same way as an increase in g. A rise in specialization costs c makes specialization more expensive, leading the principal to seek a more balanced project assignment. Finally, γ does not depend on α. Intrinsic preference does not affect the relative benefits or costs of spreading or concentrating effort across projects.

In the rest of this section, I characterize optimal project assignments across two environments: (1) heterogeneous multitasking costs and (2) heterogeneous complementarities.

4.1 Heterogeneous Multitasking Costs

Workers have varying multitasking costs r_i, but are otherwise identical i.e. $\alpha_i^p \equiv \alpha$, $c_i \equiv c$ and $g_{ij} \equiv g$. The following result demonstrates that multitasking costs are sufficient for determining workers' degree of specialization in an optimal project assignment.

Theorem 2. *In an optimal project assignment, the assignment and full utilization cutoffs σ_i and γ_i are sorted such that $r_i > r_j$ implies $\sigma_i \leq \sigma_j$ and $\gamma_i \leq \gamma_j$.*

Proof. See Appendix A.8. □

Every worker works on the most valuable project, whereas the least valuable projects are staffed only by workers who are the best at multitasking. Specialists, those who are assigned to few projects, work together on large project teams, while generalists could potentially be the *only* ones employed at a particular project.

While the extensive margin in assignments and utilization are ordered, the intensive margin is not. Effort levels are sensitive to the model parameters. On the one hand, a high multitasking cost leads the principal to reduce that worker's efforts across multiple projects, holding the set of assigned projects fixed. On the other hand, the principal seeks to assign such a worker to a *smaller* set of assigned projects, freeing up his capacity to work more. The net effect can go either way.

Example 2. There are two workers i and j and two projects p and q. The model parameters are $\alpha \equiv 1$, $c = 0.6$, $g_{ij} = 0.05$ and multitasking costs $r_i = 0.15$ and

$r_j = 0.05$. Worker i is the specialist and worker j is the generalist. I increase the principal's β weight on project p from 6 to 7 while keeping his weight on project q at 1.

	$\beta^{(p)} = 6$			$\beta^{(p)} = 7$		
	p	q	Total	p	q	Total
Worker i (specialist, $r_i = 0.15$)	1.662	$0.577^{\dagger}$	2.239	1.724	$0.329^{\dagger}$	2.053
Worker j (generalist, $r_j = 0.05$)	1.674	1.575	3.249	1.681	1.554	3.235
Gap $e_j^p - e_i^p$	$+\mathbf{0.012}$			$-\mathbf{0.044}$		

† Underutilized. Worker i is capped at project q; worker j is fully utilized everywhere.

Table 3

The intensive margin is not ordered while the extensive margin is unchanged across the two columns. The generalist is fully utilized at both projects, the specialist only at p, yet the ranking of effort levels at p reverses.

When $\beta^{(p)} = 6$, the generalists works more on p; when $\beta^{(q)} = 7$, the specialist does. The assignment pattern stays the same. The total effort column reveals why this is true. When the principal is more biased towards project p, she tightens the specialist's cap at project q. His total effort falls, dragging his marginal cost down, which frees up his capacity to work more on project p. Since the generalist is fully utilized at both projects, there is little to no room for the principal to exercise her discretion in directly choosing his efforts. Whether a specialist works more or less than a generalist on a shared project depends on the size of the parameters, and not the inherent structure of the assignment.

4.2 Heterogeneous Complementarities

I assume effort complementarity is heterogeneous across workers, but every other parameter is identical i.e. $\alpha_i^p \equiv \alpha$, $c_i \equiv c$ and $r_i \equiv r$. The natural conjecture is that central workers— those that collaborate well with many other workers— should be generalists. Placing them on many project teams amplifies efforts. However, doing so is costly because of multitasking. Assigning them to more projects weakens their contributions, and the same links that make workers central to many projects also make it expensive to assign them to many. As it turns out, the classic Katz-Bonacich centrality, which orders players' efforts in standard network games, gets the ordering wrong. Cutoffs are determined endogenously by *congestion indices*.

Definition 5. For a subset of workers $S \subseteq N$, the congestion index of worker j on **S** is:

$$\rho_j(S) := \frac{r((\mathbf{M}^S)^{-1}\mathbf{\Lambda}^S)_j}{((\mathbf{M}^S)^{-1}\mathbf{1}^S)_j}$$

where $\mathbf{M}^S$, $\mathbf{\Lambda}^S$ and $\mathbf{1}^S$ are the submatrices and subvectors with indices restricted to those in S. The ordering of workers is defined recursively. Set $N_0 := N$ and for $k = 1, ..., n$, compute

$$i(k) := \arg\max_{j \in N_{k-1}} \rho_j(N_{k-1}) \text{ where } N_k := N_{k-1} \setminus \{i(k)\}$$

Relabel worker indices in order $i(1), ..., i(n)$. Worker i's **congestion index** is given by $\rho_i := \rho_i(N_{i-1})$.

The congestion index ρ_i is a ratio of two centralities computed using $\mathbf{M}^S$ but with different weights. The denominator is the canonical Katz-Bonacich (KB) centrality $((\mathbf{M}^S)^{-1}\mathbf{1})_i$. It counts the total additional effort that comes from relaxing his incentive constraint at a project, incorporating both the direct effect for worker i and all of the complementarity effects for other workers. The numerator applies the same interaction matrix to Λ_i, which is the total shadow value of worker i's incentive constraints. It measures the opportunity cost incurred by the principal for all other projects when she increases worker i's effort towards a specific project, scaled by r. A high congestion index means it is better to concentrate that worker on only a few projects. In this manner, it is clear why traditional centrality measures do not order the assignment. A worker who has high KB centrality is also one whose incentive constraints are valuable to relax, and therefore has a large Λ_i. From the primitives alone, it is unclear which force dominates.

Importantly, the relevant network shrinks as we descend the project value ordering. Once a specialist is identified and left off projects, he is no longer a source of complementarity for any worker left at those projects. The congestion index must be recomputed for the remaining workers on the submatrix of $\mathbf{M}$ *leaving out* worker i. This is the reason for the recursive updating of N_k and why the congestion index looks very different than standard centrality measures.

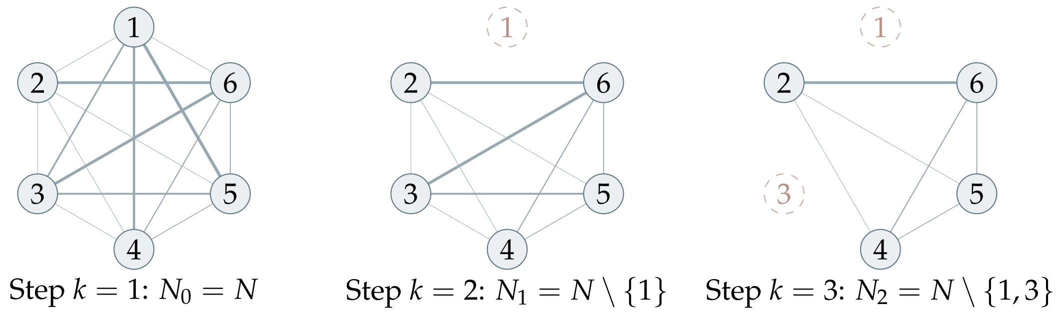


Figure 3: Once a worker is removed from a project, he stops generating complementarity to those who remain, so $\rho_i(\cdot)$ is recomputed on a strictly smaller network.

The ordering of workers can be viewed as a sequential key player problem akin t o Ballester et al. (2006). In their setting, a planner deletes a player from a network to minimize aggregate economic activity. The player has the highest *intercentrality*, which is his KB centrality but scaled to account for the activity his neighbors lose after he is removed. Here, workers are unassigned from projects when the marginal cost of keeping them on exceeds the marginal benefit. A similar KB index appears but is scaled by an endogenous object Λ_i. The difference is important because when the principal removes a worker from a particular project, his multitasking costs fall, which relaxes his incentive constraints at other projects and boosts both his and his collaborators' efforts. The numerator $r(\mathbf{M}^{-1}\mathbf{\Lambda})_i$ accounts for this redistribution effect of effort at other projects within the organization. In contrast, there is no redistribution of efforts for the removed player in Ballester et al. (2006). The congestion indices are sufficient for sorting workers.

Theorem 3. *The assignment and full utilization cutoffs σ_i and γ_i are sorted such that $\rho_i > \rho_j$ implies $\sigma_i \leq \sigma_j$ and $\gamma_i \leq \gamma_j$.*

Proof. See Appendix A.9. □

In the following example, I show how the congestion index ranking breaks away from Katz-Bonacich centrality.

	Worker j					
Worker i	1	2	3	4	5	6
1	–	0.044	0.242	0.339	0.439	0.067
2	0.044	–	0.036	0.071	0.051	0.391
3	0.242	0.036	–	0.065	0.264	0.409
4	0.339	0.071	0.065	–	0.111	0.156
5	0.439	0.051	0.264	0.111	–	0.138
6	0.067	0.391	0.409	0.156	0.138	–

Synergy matrix G, six workers.

Table 4

Example 3. There are 6 workers and 3 projects. Their synergies are given by

The other model parameters are $\alpha = 0.5$, $c = 2$, $r = 0.4$, $\beta^{(1)} = 10$, $\beta^{(2)} = 5$ and $\beta^{(3)} = 1$. Their KB centralities with respect to $\mathbf{M}$ and the projects where they are assigned and fully utilized are given by

Worker i	KB	Assignments	Fully Utilized
1	1.739	$\{1,2\}$	$\{1,2\}$
2	1.712	$\{1,2,3\}$	$\{1,2\}$
3	1.685	$\{1,2\}$	$\{1,2\}$
4	1.664	$\{1,2\}$	$\{1,2\}$
5	1.398	$\{1,2,3\}$	$\{1,2,3\}$
6	1.244	$\{1,2,3\}$	$\{1,2,3\}$

Table 5

The most central worker specializes in the top two projects, rather than being spread across multiple project teams. The worker with the second-highest centrality works on all three projects, while the workers with the third- and fourth-highest KB centrality only work on projects 1 and 2.

5 Comparative Statics

The comparative statics are governed by which constraints bind according to the assignment and full utilization patterns. I examine it across three distinct regimes, ordered by increasing principal involvement.

First, I consider full utilization and assignment, i.e., $A^p = \mathcal{F}^p = N$ for every project. Every constraint binds, and the principal's valuation plays no role. There is a full-rank linear system that describes the optimal project assignment, and its derivatives sufficiently pin down the comparative statics. Shocks decompose into a *within-project spillover* and a *network reallocation effect*. The former operates through effort complementarity while the latter captures how multitasking induces workers to redistribute efforts across multiple projects.

Second, I analyze full utilization and partial assignment i.e. $A^p = \mathcal{F}^p \subset N$ at some project. The linear system remains full-rank, but only on the support of the assignment. Shocks propagate through the organizational network induced by the assignment. Sign predictions in this regime depend on the path a shock travels through the organizational network.

Third, I consider optimal project assignments with underutilization i.e. $\mathcal{F}^p \subset A^p$ at some project. Slack incentive constraints create a rectangular linear system, and the latter is no longer sufficient for determining efforts. Interpreting binding incentive constraints as a linear budget over workers' efforts, every shock admits a Slutsky decomposition into an income effect and substitution effect. The income effect comprises the aforementioned within-project and network reallocation effects, while the substitution effect accounts for the curvature of the principal's valuation.

5.1 Full Utilization and Assignment

Under full utilization and assignment, the optimal project assignment is sufficiently determined by the binding incentive constraints and is given by $\hat{\mathbf{e}} := \mathbf{H}^{-1}\boldsymbol{\alpha}$. The following result decomposes any shock into two separate components. There is an immediate effect on a worker's incentive constraint at the shocked project governed by the within-project interaction matrix $\mathbf{M}$. The broader network reallocation effect that runs through his total effort and tightens his incentive constraints at every project is captured by the aggregate interaction matrix $\mathbf{P}$. In what follows, a positive shock induces a *local* benefit to a project but necessarily comes at a *global* cost to the organization. To describe the result, I introduce the impact vector, which measures the initial response in worker j's marginal utility at project p to a shock in a model parameter.

Definition 6. For an optimal project assignment $\mathbf{e}^*$ and a parameter $\theta \in \{\alpha_j^q, c_j, r_j, g_{jk}\}$, $\mathbf{d}^p(\theta) \in \mathbb{R}^n$ is the **impact vector at project** p and has entries

$$\mathbf{d}_i^p(\theta) := \frac{\partial}{\partial \theta}\left(\alpha_i^p + \sum_{j \neq i} g_{ij} e_j^{p*} - c_i e_i^{p*} - r_i \sum_{q \neq p} e_i^{q*}\right)$$

The impact vectors are

$$\begin{aligned} \mathbf{d}^p(\alpha_j^q) &= \delta_{pq}\boldsymbol{\epsilon}_j & \mathbf{d}^p(c_j) &= -e_j^{p*}\boldsymbol{\epsilon}_j \\ \mathbf{d}^p(g_{jk}) &= e_k^{p*}\boldsymbol{\epsilon}_j + e_j^{p*}\boldsymbol{\epsilon}_k & \mathbf{d}^p(r_j) &= -(E_j^* - e_j^{p*})\boldsymbol{\epsilon}_j \end{aligned}$$

where δ_{pq} is the Kronecker delta, $\boldsymbol{\epsilon}_j$ is the j-th standard basis vector and E_j^* is worker j's total effort. The shock decomposition is pinned down by how the impulse response, captured by the impact vector, propagates throughout a project and the entire organization.

Proposition 6. *In an optimal project assignment with full utilization and assignment, the following holds for every parameter θ and every project p*

$$\frac{\partial \mathbf{e}^{p*}}{\partial \theta} = \underbrace{\mathbf{M}^{-1}\mathbf{d}^p(\theta)}_{\textit{within-project spillover}} - \underbrace{\mathbf{M}^{-1}\mathbf{R}\mathbf{P}^{-1}\sum_{q=1}^m \mathbf{d}^q(\theta)}_{\textit{network reallocation}}$$

Moreover $\frac{\partial e_i^{p*}}{\partial \alpha_j^p} > 0$ for all j and $\frac{\partial e_i^{p*}}{\partial \alpha_j^q} < 0$ for all $q \neq p$.

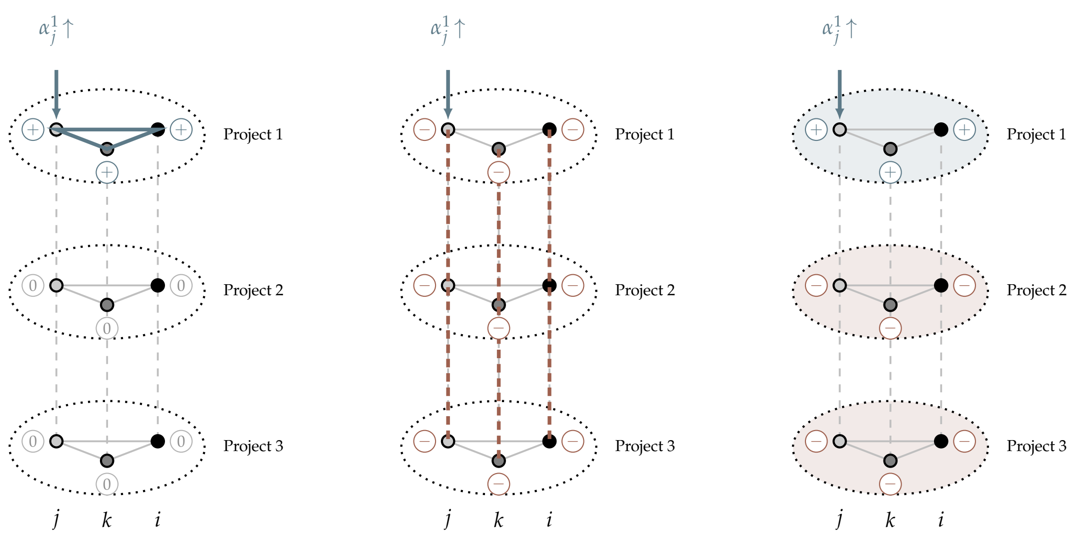
Proof. See Appendix A.10. □

The inverse matrices $\mathbf{M}^{-1}$ and $\mathbf{P}^{-1}$, which are both nonnegative matrices, play a starring role in the comparative statics. To see the mechanics behind the shock decompositions, I focus on the comparative static for α_j^q.

$$\frac{\partial e_i^{p*}}{\partial \alpha_j^q} = \underbrace{\mathbf{M}_{ij}^{-1}\delta_{pq}}_{\text{within-project spillover}} - \underbrace{\sum_{\ell=1}^n \mathbf{M}_{i\ell}^{-1} r_\ell \mathbf{P}_{\ell j}^{-1}}_{\text{network reallocation}}$$

If $q = p$, then worker j is more willing to work on project p, and complementarity makes everyone else supply more effort at p. In response, the principal raises every worker's effort cap at project p. This is captured by the (i, j)-element of the within-project interaction matrix $\mathbf{M}^{-1}$.

The second term represents a separate reallocation channel. Recall from Section 2.3 that if all incentive constraints bind, the total effort vector satisfies $\mathbf{E} = \mathbf{P}^{-1} \sum_{p=1}^{m} \boldsymbol{\alpha}^p$. Therefore, the entries of $\mathbf{P}^{-1}$ detail how a worker's *total effort* responds to a preference shock. Now consider worker ℓ's response to an increase in α_j^p. He raises his effort at project p, which raises his total effort and, with it, his marginal cost by r_ℓ. It is now more expensive for him to exert effort towards *any* project. Consequently, he *lowers* his effort on project p, effectively reducing the effort complementarities received by worker i at project p. The $\mathbf{M}_{i\ell}^{-1}$ entry captures this effect, and the summation accounts for all workers also working on project p. The network reallocation effect causes a drop in quality for *every* project not subject to the positive preference shock.



(a) Within-project spillover. (b) Network reallocation. (c) Net response.

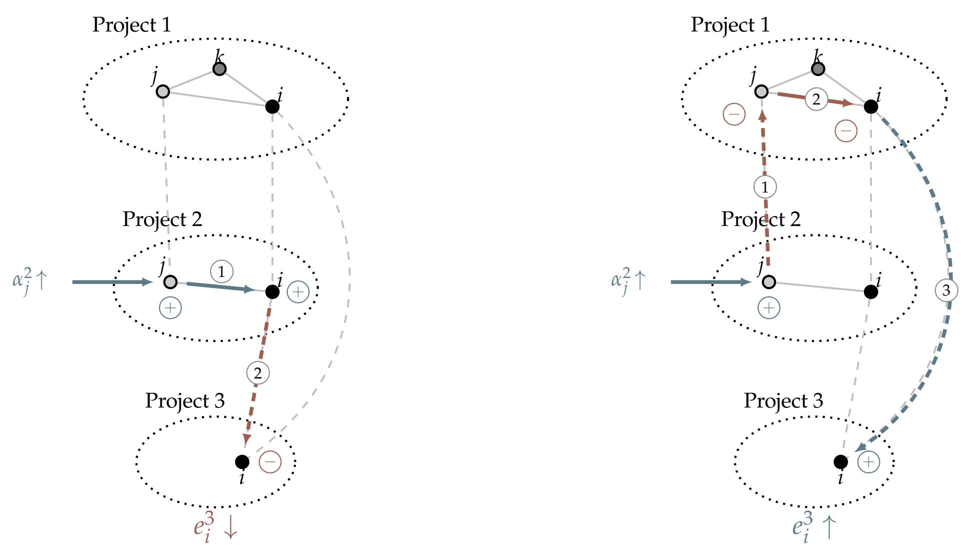
Figure 4: Nodes of the same shade correspond to the same worker while the dotted ellipses represent different projects. Worker j receives a preference shock at project 1. The within-project spillover in (a) travels along the solid edges within the shocked ellipse alone. The network reallocation in (b) travels down the dashed edges: higher effort raises each worker's total effort and his marginal cost by r_ℓ, causing him to reduce effort at every project. Adding (a) and (b) gives (c): a positive shock local to one project degrades the quality of all others.

Cost shocks behave a bit differently. An increase in c_j or r_j necessarily reduces workers' total efforts, but for a project, the two effects push in opposite directions. Lowering total efforts also lowers a worker's marginal cost, inducing him to increase efforts at a project. The sign prediction depends on the exact magnitudes of the efforts in the optimal project assignment.

5.2 Full Utilization and Partial Assignment

By leaving some workers off some projects, shock transmission is now influenced by the pattern of active project assignments. The principal cannot anticipate the direction a preference shock goes in as it depends on the staffing pattern.

Example 4. Suppose there are three projects $(1, 2, 3)$ and three workers (i, j, k), and the optimum project assignment has the following nested structure: worker i is assigned to all 3 projects, worker j is assigned to projects 1 and 2 and worker k is assigned to project 1.



(a) The direct route $(j, 2) \to (i, 2) \to (i, 3)$. Complementarity raises worker i's effort at project 2; his multitasking cost then pulls him off project 3.

(b) The indirect route, $(j, 2) \to (j, 1) \to (i, 1) \to (i, 3)$. Worker j reduces effort at project 1, worker i loses complementarity, and he reduces effort at project 1. This frees up effort for i to spend at project 3.

Figure 5: Two routes for the same shock under partial assignment.

Consider an increase in α_j^2. Worker i raises his efforts at project 2 due to an increase in complementarity, and cuts back on his efforts at project 3. There is another route the shock travels. Notice that worker j withdraws from project 2, and so worker i follows suit at project 2, saving capacity to work on his other projects. This effect pushes worker j to raise his effort at project 3. Ambiguity in the direction of the shock depends on the strengths of his complementarity. For example, if g_{ij} is large, then his gain at project 2 dominates, leading to higher total effort and a contraction at project 3. On the other hand, if g_{ij} is low, his effort reduction at project 1 dominates, and he is available to work more on project 3. In contrast to the full assignment case, a positive preference shock for one project could *raise* the quality of another project.

Under partial assignment, the impact vectors remain the same. The primary difference is that $\mathbf{M}^{(A^p)}$, the principal submatrix of $\mathbf{M}$ restricted to workers assigned to p, regulates within-project spillovers for project p. The network reallocation effect changes as well because each project carries a different $\mathbf{M}^{(A^p)}$ matrix. The operator that captures how total efforts respond to the shock is no longer $\mathbf{P}$, but $\mathcal{P}$ built from the individual $\mathbf{M}^{(A^p)}$ matrices. It has entries

$$\mathcal{P}_{ij} = \delta_{ij} + r_j \sum_{p \in A_i \cap A_j} (\mathbf{M}^{(A^p)})^{-1}_{ij}.$$

Its inverse $\mathcal{P}^{-1}$ fails to be nonnegative, and now workers' total effort responses could move in either direction.

Theorem 4. *In an optimal project assignment with full utilization and partial assignment, the following holds for every parameter θ and every project p*

$$\frac{\partial e^{p*}}{\partial \theta} = \underbrace{(\mathbf{M}^{(A^p)})^{-1}\mathbf{d}^p(\theta)}_{\textit{within-project spillover}} - \underbrace{(\mathbf{M}^{(A^p)})^{-1}\mathbf{R}\big(\mathcal{P}^{-1}\boldsymbol{\sigma}(\theta)\big)}_{\textit{network reallocation}}$$

where the right-hand vector $\boldsymbol{\sigma}$ has entries

$$\sigma_i(\theta) := \sum_{p\,:\,i \in A^p} \big((\mathbf{M}^{(A^p)})^{-1}\mathbf{d}^p(\theta)\big)_i$$

Proof. See Appendix A.11. □

The comparative statics are carried by the assignment-specific interaction matrices and the managerial insight is different. Under full assignment and utilization, a positive preference shock raised quality at the project subject to the shock and lowered qualities everywhere else.[15] As the example demonstrated, a positive preference shock could raise the qualities of other projects; imbalanced project assignments lead to different routes the shock can take.

5.3 Underutilization

When the optimal project assignment features underutilization, binding incentive constraints do not determine optimal effort levels. The principal has discretion in choosing efforts; slack in incentive constraints allows the principal to substitute one worker's effort for another. To recover a Slutsky decomposition, I interpret a binding constraint as a budget for effort.

$$\alpha_i^p = c_i e_i^p + r_i \sum_{q \in P_i, q \neq p} e_i^q - \sum_{j \in A^p, j \neq i} g_{ij} e_j^p$$

α_i^p is the income of effort available to the principal, c_i is the own price of worker i's effort at project p, r_i is the price of his effort at other projects and $-g_{ij}$ is a discount because other workers' efforts make it cheaper for worker i to work more. The price movements for worker i's effort at project p are all conveniently recorded in the (i, p) column of the system interaction matrix $\mathbf{H}$. The following vector measures how sensitive the principal is to price changes in the optimum, where slack budgets do not matter because relaxing them does not contribute marginal value.

Definition 7. For an optimal project assignment with multipliers $\boldsymbol{\lambda}^*$, let $\mathcal{F} := \{(i, p) \mid \lambda_i^{p*} > 0\}$. For a parameter $\theta \in \{\alpha_j^q, c_j, r_j, g_{jk}\}$, $\boldsymbol{\pi}(\theta) \in \mathbb{R}^{mn}$ is the **price vector** and has entries

$$\pi_i^p(\theta) := \frac{\partial}{\partial \theta} \left[\mathbf{H}\boldsymbol{\lambda}^*\right]_{(i,p)} = \sum_{(j,q) \in \mathcal{F}} \lambda_j^{q*} \frac{\partial \mathbf{H}[(j, q), (i, p)]}{\partial \theta},$$

[15] One can readily verify that Proposition 6 is a special case of the above: under full assignment $(\mathbf{M}^{(A^p)})^{-1} \equiv \mathbf{M}$, so $\mathcal{P} = \mathbf{I} + m\mathbf{M}^{-1}\mathbf{R} = \mathbf{M}^{-1}\mathbf{P}$ and $\sigma(\theta) = \mathbf{M}^{-1} \sum_{q=1}^{m} \mathbf{d}^q(\theta)$.

The price vectors for each parameter are

$$\boldsymbol{\pi}(\alpha_j^q) = \mathbf{0} \qquad\qquad \boldsymbol{\pi}(c_j) = \sum_{p=1}^{m} \lambda_j^{p*} \boldsymbol{\epsilon}_{(j,p)}$$

$$\boldsymbol{\pi}(g_{jk}) = -\sum_{p=1}^{m} (\lambda_k^{p*} \boldsymbol{\epsilon}_{(j,p)} + \lambda_j^{p*} \boldsymbol{\epsilon}_{(k,p)}) \qquad\qquad \boldsymbol{\pi}(r_j) = \sum_{p=1}^{m} (\sum_{q \neq p} \lambda_j^{q*}) \boldsymbol{\epsilon}_{(j,p)}$$

Now I proceed with defining the substitution and income matrices that make up the Slutsky decomposition.

Definition 8. Define the Jacobian and its inverse (if it exists) in corresponding block matrix form

$$\mathbf{J} := \begin{bmatrix} \nabla_{\mathbf{e}}^2 V & \mathbf{H}_{\mathcal{F}} \\ \mathbf{H}_{\mathcal{F}}^{\intercal} & \mathbf{0} \end{bmatrix} \qquad \mathbf{J}^{-1} := \begin{bmatrix} (\mathbf{J}^{-1})_{11} & (\mathbf{J}^{-1})_{12} \\ (\mathbf{J}^{-1})_{21} & (\mathbf{J}^{-1})_{22} \end{bmatrix}$$

where $\mathbf{H}_{\mathcal{F}}$ is the submatrix of $\mathbf{H}$ restricted to indices in $\mathcal{F}$. The **substitution matrix** is $\mathbf{S} := (\mathbf{J}^{-1})_{11}$ and the **income matrix** is $\mathbf{Y} := (\mathbf{J}^{-1})_{12}$.

Let $\mathcal{A} := \{(i,p) \mid e_i^{p*} > 0\}$ be the set of all worker-project pairs that are assigned and consider an effort vector $\mathbf{x} \in \mathbb{R}^{\mathcal{A}}$. For $\mathbf{x}$ to be a reallocation of effort that leaves every binding incentive constraint exactly binding, it must satisfy $\mathbf{H}_{\mathcal{F}}^{\intercal}\mathbf{x} = \mathbf{0}$. It is the effort analog of a movement along a budget line in consumer theory. Notice that the substitution matrix $\mathbf{S}$, which is symmetric, describes these types of movements because it satisfies $\mathbf{H}_{\mathcal{F}}^{\intercal}\mathbf{S} = \mathbf{0}$. Moreover, its diagonal is nonpositive because it is negative semidefinite, reflecting the law of compensated demand. The income matrix $\mathbf{Y}$ satisfies $\mathbf{H}_{\mathcal{F}}^{\intercal}\mathbf{Y} = \mathbf{I}$ and so it does no capture any reallocative direction. Together, $\mathbf{S}$ and $\mathbf{Y}$ comprise the Slutsky decomposition.

Theorem 5. *If $\nabla_e^2 V$ is strictly negative definite on $\ker(\mathbf{H}_{\mathcal{F}}^{\top})$, then $\mathbf{J}^{-1}$ exists and for every parameter θ,*

$$\frac{\partial \mathbf{e}^*}{\partial \theta} = \underbrace{\mathbf{S}\,\boldsymbol{\pi}(\theta)}_{\textit{substitution}} + \underbrace{\mathbf{Y}\,\mathbf{d}_{\mathcal{F}}(\theta)}_{\textit{income}},$$

where $\mathbf{d}_{\mathcal{F}}(\theta)$ is the impact vector restricted to $\mathcal{F}$.

Proof. See Appendix A.12. □

The theorem requires a technical condition: the optimum has to uniquely pin down individual efforts so that the formulas hold by applying the Implicit Function Theorem. Geometrically, the incentive constraints form a polytope, and underutilization means optimal efforts sit on a face of the polytope. This is exactly why the substitution effect was nonexistent in Proposition 6 and Theorem 4; if every incentive constraint binds, then the optimal project assignment resides at a vertex of the polytope. In other words, the preceding theorems only exhibit income effects.

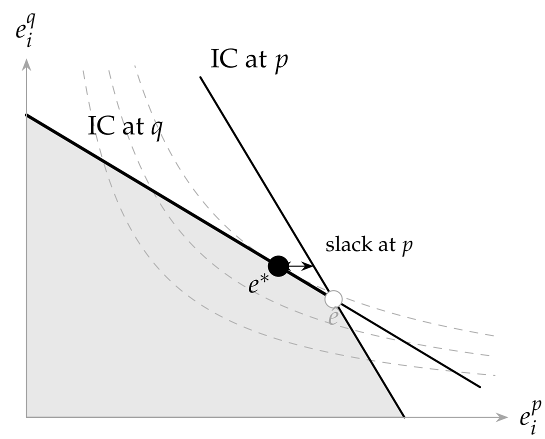


Figure 6: Underutilization at $\mathbf{e}^*$ for one worker and two projects.

The formula for a preference shock, which is the income effect, is embedded in the other comparative statics. To contrast this Slutsky equation with that of traditional consumer theory, consider an increase in specialization cost for worker i:

$$\frac{\partial \mathbf{e}^*}{\partial c_i} = \sum_{p \le \gamma_j} \left(\lambda_j^{p*} \mathbf{S}_{\cdot,(j,p)} - e_j^{p*} \frac{\partial \mathbf{e}^*}{\partial \alpha_i^p} \right)$$

The principal, who is the consumer here, faces a budget per binding incentive constraint instead of just a single budget. The income effect is weighted by the effort whose price moved but the sum runs over multiple projects because one price change affects several incentive constraints at once. Similarly, the substitution effect is weighted by λ_j^{p*} per project p.

6 Discussion

6.1 Effort Caps and Contracts

By studying optimal project assignments via effort caps in isolation, I prohibit the principal from contracting on effort. A natural question is whether she could achieve the outcomes if transfers were allowed. The comparison here is about implementable outcomes, not which instrument the principal would prefer as the main analysis concerns environments where contracting is unavailable. Answering the latter requires additional assumptions on how her valuation of projects relates to the costs of paying her workers.

I show that effort caps implement effort profiles that contracts which reward effort cannot.

Proposition 7. *Suppose worker i's payoff under contract w_i is given by $U_i(\mathbf{e}_i, \mathbf{e}_{\text{-}i}) + w_i(\mathbf{e}_i, \mathbf{e}_{\text{-}i})$ where w_i is nondecreasing in each e_i^p. No effort profile where worker i is underutilized at project p is an equilibrium.*

Proof. See Section A.13. □

Equivalently, any contract implementing a profile with underutilization for worker i at project p must be strictly decreasing in e_i^p. For an illustration, consider a linear contract for each worker i

$$w_i(\mathbf{e}_i) = \sum_{p=1}^{m} b_i^p e_i^p \text{ where } b_i^p \in \mathbb{R}.[16]$$

To implement an effort profile, the required rate is the negative of the worker's marginal utility at the target. To underutilize a worker and leave his incentive constraint slack, the principal must fine him for working. Effort caps and *nonnegative* wages are different instruments that move effort in the opposite direction. A cap holds a worker back on a project he would like to contribute more to, while a posi-

[16] I leave out a fixed payment because it does not affect workers' incentives to exert effort. It does, however, affect whether limited liability binds and can be chosen so that total pay is always nonnegative.

tive wage pushes him to do so. Instead, contracts can achieve effort profiles where worker marginal utilities are negative, something effort caps cannot accomplish.

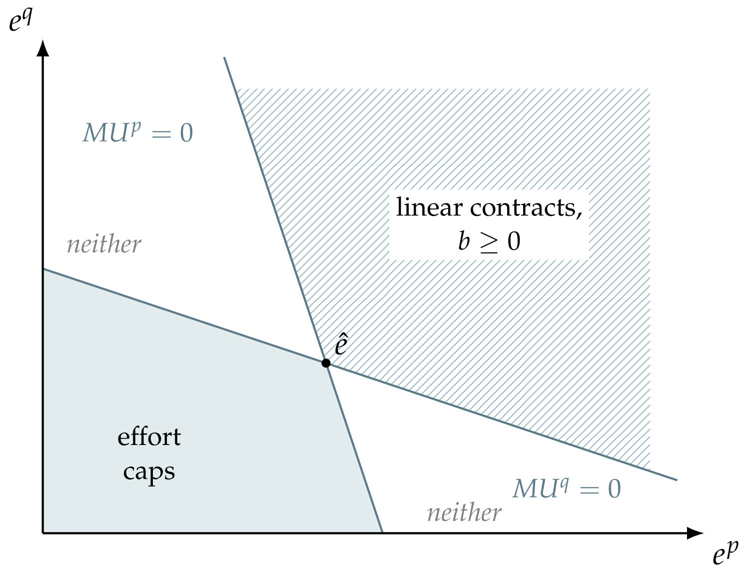


Figure 7: Effort profiles implementable by caps and by contracts, for one worker and two projects. The lines are the worker's incentive constraints. The two sets meet only at the unconstrained equilibrium $\hat{e}$, where every marginal utility vanishes. In the two unshaded regions, neither instrument alone suffices.

6.2 Asymmetric Complementarities

I imposed symmetric complementarities early on to write the workers' effort game as an exact potential game, which buys us uniqueness of equilibrium. Beyond that, complementarities can be asymmetric, and all of the main results still hold.

It is imperative that worker interactions are complements and not substitutes within a project. If interactions are substitutes i.e. $g_{ij} < 0$, then the team-coordinated effort transfer argument is no longer available. Shifting a worker's effort from a low-value project to a high-value one encourages other workers to desire more work at the former, rendering the arbitrage moot. The nested hierarchy structure no longer holds.

6.3 Effort Budgets

As the leading examples in the introduction suggested, firms and organizations typically allocate the *share* of effort workers have across different projects. That is, there is a ceiling to the effort the principal can implement. This restriction places an additional constraint in the principal's convex optimization problem where, for all workers i, $\sum_{p=1}^{m} e_i^p \leq T_i$, where T_i is an exogenously given hard limit. The main result still holds because each external budget, similar to other objects like total effort, is worker-specific and plays no role in the analysis when we compare any two projects. The results in the parametric Cobb-Douglas setting change. The budget is a second sorting force that competes with multitasking costs, and whether workers are designated specialists or generalists depends on how lax their effort budget constraint is.

7 Concluding Remarks

Organizations that do not pay for effort on a day-to-day basis regularly choose how much they want implemented. Effort caps are not a substitute for incentives and deserve their own analysis. It is essentially two instruments in one: it selects which workers collaborate with whom, and it dictates the breadth of what a worker works on in the organization. Optimally assigning workers to projects by setting effort caps turns out to be far more simple than the richness of the model suggests. A single endogenous ranking of projects sorts every worker's schedule into three consecutive intervals— fully utilized, assigned but underutilized, unassigned (Theorem 1). Workers tend to be underutilized at projects they most prefer. Because the assignment selects the network of interactions, an organization's response to shocks is implicitly designed by the principal. A shock favorable to one project could end up being bad news for other projects in the organization, depending on the pattern of assignment and utilization in the optimum.

There are many promising extensions for future research. Because this paper studies assignment in isolation, a natural next step is to study the joint design of assignments and incentives, especially given that they affect effort in opposite ways. Project assignments could also be studied in tandem with career concerns, as success on certain projects could advance a worker's career more than others. Lastly, Assumption 1 guarantees a unique equilibrium and nullifies equilibrium selection concerns. Relaxing it allows us to study which project assignments remain robust when the worst equilibrium is played. I hope that this work inspires future research in studying staffing decisions and project assignment through the lens of a network design problem.

References

Alonso, Ricardo, and Niko Matouschek. 2008. "Optimal Delegation." The Review of Economic Studies, 75(1): 259–293.

Ambrus, Attila, Eduardo M. Azevedo, and Yuichiro Kamada. 2013. "Hierarchical cheap talk." Theoretical Economics, 8(1): 233–261.

Armstrong, Mark, and John Vickers. 2010. "A Model of Delegated Project Choice." Econometrica, 78(1): 213–244.

Bala, Venkatesh, and Sanjeev Goyal. 2000. "A Noncooperative Model of Network Formation." Econometrica, 68(5): 1181–1229.

Ballester, Coralio, Antoni Calvó-Armengol, and Yves Zenou. 2006. "Who's Who in Networks. Wanted: The Key Player." Econometrica, 74(5): 1403–1417.

Becker, Gary S. 1973. "A Theory of Marriage: Part I." Journal of Political Economy, 81(4): 813–846.

Belhaj, Mohamed, and Frédéric Deroïan. 2018. "Targeting the key player: An incentive-based approach." Journal of Mathematical Economics, 79: 57–64.

Benson, Alan M., and Kathryn L. Shaw. 2025. "What Do Managers Do? An Economist's Perspective."

Bramoullé, Yann, and Rachel Kranton. 2007. "Public goods in networks." Journal of Economic Theory, 135(1): 478–494.

Bramoullé, Yann, Rachel Kranton, and Martin D'Amours. 2014. "Strategic Interaction and Networks." American Economic Review, 104(3): 898–930.

Carpenter, Jeffrey, and Erick Gong. 2016. "Motivating Agents: How Much Does the Mission Matter?" Journal of Labor Economics, 34(1): 211–236.

Chen, Ying-Ju, Yves Zenou, and Junjie Zhou. 2018. "Multiple Activities in Networks." American Economic Journal: Microeconomics, 10(3): 34–85.

Che, Yeon-Koo, and Seung-Weon Yoo. 2001. "Optimal Incentives for Teams." American Economic Review, 91(3): 525–541.

Chiocchio, François, Daniel Forgues, David Paradis, and Ivanka Iordanova. 2011. "Teamwork in Integrated Design Projects: Understanding the Effects of Trust, Conflict, and Collaboration on Performance." Project Management Journal, 42(6): 78–91.

Claveria-Mayol, Marc, Pau Milan, and Nicolas Oviedo-Davila. 2024. "Incentive Contracts and Peer Effects in the Workplace *."

Cottle, Richard W., Jong-Shi Pang, and Richard E. Stone. 2009. The Linear Complementarity Problem. Philadelphia, Pa:Society for Industrial and Applied Mathematics.

Coviello, Decio, Andrea Ichino, and Nicola Persico. 2014. "Time Allocation and Task Juggling." American Economic Review, 104(2): 609–623.

Dasaratha, Krishna, Benjamin Golub, and Anant Shah. 2025. "Incentive Design With Spillovers." EC '25, 505. New York, NY, USA:Association for Computing Machinery.

Demange, Gabrielle. 2025. "Dual communication in a social network: Contributing and dedicating attention." Games and Economic Behavior, 153: 359–385.

Dessein, Wouter, Luis Garicano, and Robert Gertner. 2010. "Organizing for Synergies." American Economic Journal: Microeconomics, 2(4): 77–114.

Dewatripont, Mathias, Ian Jewitt, and Jean Tirole. 2000. "Multitask agency problems: Focus and task clustering." European Economic Review, 44(4): 869–877.

Franke, Jörg, and Tahir Öztürk. 2015. "Conflict networks." Journal of Public Economics, 126: 104–113.

Frankel, Alexander. 2014. "Aligned Delegation." American Economic Review, 104(1): 66–83.

Freund, Lukas B. 2025. "Superstar Teams: The Micro Origins and Macro Implications of Coworker Complementarities."

Fu, Richard, Ajay Subramanian, and Anand Venkateswaran. 2016. "Project Characteristics, Incentives, and Team Production." Management Science, 62(3): 785–

801.

Galeotti, Andrea, Benjamin Golub, and Sanjeev Goyal. 2020. "Targeting Interventions in Networks." Econometrica, 88(6): 2445–2471.

Galeotti, Andrea, Benjamin Golub, Sanjeev Goyal, Omer Tamuz, and Eduard Talamas. 2025. "Robust Market Interventions." EC '25, 385. New York, NY, USA:Association for Computing Machinery.

Geanakoplos, John, and Paul Milgrom. 1991. "A theory of hierarchies based on limited managerial attention." Journal of the Japanese and International Economies, 5(3): 205–225.

Georgiadis, George. 2015. "Projects and Team Dynamics." The Review of Economic Studies, 82(1): 187–218.

Glover, Jonathan, and Eunhee Kim. 2021. "Optimal Team Composition: Diversity to Foster Implicit Team Incentives." Management Science, 67(9): 5800–5820.

Hart, Oliver, and John Moore. 2005. "On the Design of Hierarchies: Coordination versus Specialization." Journal of Political Economy, 113(4): 675–702.

Holmstrom, Bengt, and Paul Milgrom. 1991. "Multitask Principal–Agent Analyses: Incentive Contracts, Asset Ownership, and Job Design." The Journal of Law, Economics, and Organization, 7: 24–52.

Itoh, Hideshi. 1991. "Incentives to Help in Multi-Agent Situations." Econometrica, 59(3): 611–636.

Jackson, Matthew O., and Asher Wolinsky. 1996. "A Strategic Model of Social and Economic Networks." Journal of Economic Theory, 71(1): 44–74.

Kremer, Michael. 1993. "The O-Ring Theory of Economic Development." The Quarterly Journal of Economics, 108(3): 551–575.

Kristof-Brown, Amy L., Ryan D. Zimmerman, and Erin C. Johnson. 2005. "Consequences of Individuals' Fit at Work: A Meta-Analysis of Person–Job, Person–Organization, Person–Group, and Person–Supervisor Fit." Personnel Psychology, 58(2): 281–342.

Leroy, Sophie. 2009. "Why is it so hard to do my work? The challenge of attention residue when switching between work tasks." Organizational Behavior and Human Decision Processes, 109(2): 168–181.

Mas, Alexandre, and Enrico Moretti. 2009. "Peers at Work." American Economic Review, 99(1): 112–145.

Minni, Virginia. 2026. "Making the Invisible Hand Visible: Managers and the Allocation of Workers to Jobs." The Quarterly Journal of Economics, 141(3): 1871–1920.

Prat, Andrea. 2002. "Should a team be homogeneous?" European Economic Review, 46(7): 1187–1207.

Rosen, J. B. 1965. "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games." Econometrica, 33(3): 520–534.

Sadler, Evan, and Steve Yeh. 2026. "A Shocking Reversal: Flipping Complements and Substitutes in Network Games." Working Paper.

Visser, Bauke. 2000. "Organizational communication structure and performance." Journal of Economic Behavior & Organization, 42(2): 231–252.

Wheelwright, Steven C., and Kim B. Clark. 1992. Revolutionizing Product Development: Quantum Leaps in Speed, Efficiency, and Quality. Simon and Schuster.

Xu, Jin, Yves Zenou, and Junjie Zhou. 2022. "Equilibrium characterization and shock propagation in conflict networks." Journal of Economic Theory, 206: 105571.

Yeh, Steve. 2026. "Optimal Project Task Assignments." Working Paper.

Zenou, Yves, and Junjie Zhou. 2026. "Games on Multiplex Networks." American Economic Review, 116(4): 1415–1458.

A Omitted Proofs

A.1 Proof of Proposition 1

(Forward Direction). Assume that $\mathbf{e}$ is a Nash equilibrium given $\bar{\mathbf{e}}$. If $e_i^p < \bar{e}_i^p$, then

$$\frac{\partial U_i}{\partial e_i^p} = 0 \implies \alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p = c_i e_i^p + r_i \sum_{q \neq p} e_i^q$$

If $e_i^p = \bar{e}_i^p$, and worker i's marginal utility at project p is negative, then he profits from reducing his efforts, contradicting that he is in a Nash equilibrium. Because worker utility is concave in own efforts, the first-order conditions are necessary and sufficient.

(Reverse Direction). I show that $\mathbf{e}$ with $\alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p \geq c_i e_i^p + r_i \sum_{q \neq p} e_i^q$ whenever $e_i^p > 0$ is an equilibrium with $\bar{\mathbf{e}} = \mathbf{e}$. Because worker utilities are concave in own efforts, a weakly positive marginal utility at any boundary point means that reducing effort cannot be a profitable deviation. Therefore e_i^p is a best response for worker i at project p given effort caps $\bar{\mathbf{e}}$. □

A.2 Proof of Lemma 1

Suppose an effort profile satisfies every incentive constraint. Fix a project p and stack each worker's incentive constraint:

$$\alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p \geq c_i e_i^p + r_i \sum_{q \neq p} e_i^q \implies \boldsymbol{\alpha}^p \geq (\mathbf{C} - \mathbf{R} - \mathbf{G})\mathbf{e}^p + \mathbf{R}\mathbf{E}$$

where $\mathbf{E} := [\sum_{p=1}^m e_1^p \cdots \sum_{p=1}^m e_n^p]$ is the vector of worker total efforts. The implication holds by rewriting $\sum_{q \neq p} e_i^q = \sum_{q=1}^m e_i^q - e_i^p$. Finally, aggregate the incentive constraints across all m projects to obtain an inequality in total efforts

$$\sum_{p=1}^m \boldsymbol{\alpha}^p \geq (\mathbf{C} - \mathbf{R} - \mathbf{G})\mathbf{E} + m\mathbf{R}\mathbf{E} \implies \sum_{p=1}^m \boldsymbol{\alpha}^p \geq \mathbf{P}\mathbf{E}.$$

Assumption 1 and complementarity between workers guarantee $\mathbf{P}$ has a nonnegative inverse matrix $\mathbf{P}^{-1} \geq \mathbf{0}$. This is because $\mathbf{P}$ is positive definite: for any $\mathbf{u} \neq \mathbf{0}$, $m\mathbf{u}^\top \mathbf{P}\mathbf{u} = (\mathbf{u} \otimes \mathbf{1}_m)^\top \mathbf{H}(\mathbf{u} \otimes \mathbf{1}_m) > 0$. It also has nonpositive off-diagonals $-g_{ij}$, and so it is a symmetric M-matrix. Having a nonnegative inverse matrix is a property of M-matrices. □

A.3 Proof of Proposition 2

Assume that $r_i \tilde{E}_i \leq \min_p \alpha_i^p$. Because $\mathbf{P}$ has a nonnegative diagonal and has negative off-diagonals and has positive eigenvalues by Assumption 1, it is an M-matrix and $\tilde{\mathbf{E}} > 0$. By construction, it satisfies

$$(c_i + r_i(m-1))\tilde{E}_i = \sum_{p=1}^{m} \alpha_i^p + \sum_{j \neq i} g_{ij} \tilde{E}_j$$

Let $\mathcal{A}_i^{(k)}$ be the set of the k largest entries of $\{\alpha_i^p\}_{p=1}^m$. Then subtract to obtain

$$(c_i + r_i(m-1))\tilde{E}_i - \sum_{p \in \mathcal{A}_i^{(k)}} \alpha_i^p - \sum_{j \neq i} g_{ij} \tilde{\mathbf{E}}_j = \sum_{p \notin \mathcal{A}_i^{(k)}} \alpha_i^p \geq (m-k) \min_p \alpha_i^p \geq (m-k) r_i \tilde{E}_i$$

Rearrange to get

$$(m-k) r_i \tilde{E}_i \leq (c_i + r_i(m-1))\tilde{E}_i - \sum_{p \in \mathcal{A}_i^{(k)}} \alpha_i^p - \sum_{j \neq i} g_{ij} \tilde{E}_j$$

$$\implies (c_i + r_i(k-1))\tilde{E}_i \geq \sum_{p \in \mathcal{A}_i^{(k)}} \alpha_i^p + \sum_{j \neq i} g_{ij} \tilde{E}_j \quad (*)$$

Now I show that for any implementable $\mathbf{e}$, the total efforts $E_i := \sum_{p=1}^m e_i^p$ are bounded from above by $\tilde{E}_i$. Summing worker i's incentive constraints over projects $p \in P_i$ gives

$$\alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p \geq c_i e_i^p + r_i \sum_{q \neq p} e_i^q \implies \sum_{p \in P_i} \alpha_i^p + \sum_{j \neq i} g_{ij} \sum_{p \in P_i} e_j^p \geq \sum_{p \in P_i} (c_i e_i^p + r_i \sum_{q \neq p} e_i^q)$$

The cost side simplifies to

$$\sum_{p \in P_i} (c_i e_i^p + r_i \sum_{q \neq p} e_i^q) = (c_i + r_i(|P_i| - 1))E_i$$

Now suppose that $\zeta := \max_i(E_i/\tilde{E}_i) > 1$. Then multiply and divide by $\tilde{E}_i$ on both the left and right sides to get

$$\zeta(c_i + r_i(|P_i| - 1))\tilde{E}_i \leq \sum_{p \in P_i} \alpha_i^p + \zeta \sum_{j \neq i} g_{ij}\tilde{E}_j = \sum_{p \in P_i} \alpha_i^p + \sum_{j \neq i} g_{ij}\tilde{E}_j + (\zeta - 1)\sum_{j \neq i} g_{ij}\tilde{E}_j$$

Applying the $(*)$ relation, we have

$$\sum_{p \in P_i} \alpha_i^p + \sum_{j \neq i} g_{ij}\tilde{E}_j + (\zeta - 1)\sum_{j \neq i} g_{ij}\tilde{E}_j \leq (c_i + r_i(|P_i| - 1))\tilde{E}_i + (\zeta - 1)\sum_{j \neq i} g_{ij}\tilde{E}_j$$

Chaining the inequality from the two lines above:

$$(\zeta - 1)(c_i + r_i(|P_i| - 1))\tilde{E}_i \leq (\zeta - 1)\sum_{j \neq i} g_{ij}\tilde{E}_j \implies (c_i + r_i(|P_i| - 1))\tilde{E}_i \leq \sum_{j \neq i} g_{ij}\tilde{E}_j$$

This implies that $\sum_{p \in \mathcal{A}_i^{(k)}} \alpha_i^p \leq 0$, which is a direct contradiction that $\alpha_i^p > 0$ for all i and p.

Now assume $\mathbf{e}$ is implementable with $e_i^p = 0$. Then $\sum_{q \neq p} e_i^q = \mathcal{E}_i \leq \tilde{\mathbf{E}}_i$ and so

$$\alpha_i^p + \sum_{j \neq i} g_{ij} e_j^p \geq \min_p \alpha_i^p \geq r_i \tilde{\mathbf{E}}_i \geq r_i \sum_{q \neq p} e_i^q$$

Because worker i's incentive constraint at p is always satisfied given zero effort e_i^p, every implementable profile must therefore satisfy every incentive constraint and the feasible set of the mixed-integer nonlinear program in Appendix B coincides with that of the convex program. □

A.4 Proof of Theorem 1

The stationarity condition for the Lagrangian of the principal's optimization problem with respect to e_i^p is

$$\frac{\partial V}{\partial \mathcal{E}^p} + \sum_{j \neq i} g_{ij}\lambda_j^p - c_i\lambda_i^p - r_i \sum_{q \neq p} \lambda_i^q + \mu_i^p = 0$$

Written in terms of worker i's sum of multipliers on the incentive constraints $\Lambda_i := \sum_{p=1}^m \lambda_i^p$ is:

$$\frac{\partial V}{\partial \mathcal{E}^p} + \sum_{j \neq i} g_{ij}\lambda_j^p + (r_i - c_i)\lambda_i^p - r_i\Lambda_i + \mu_i^p = 0$$

Our goal is to understand the structure of λ_i^p and μ_i^p in the optimum. I first observe that the strict slackness condition where $e_i^p = 0$ implies $\lambda_i^p = 0$ is a generic condition of the optimum. Appendix C details the full argument why it is so. I stress that the converse is *not* generically true, as the principal underutilizes workers at projects she assigns them to. Using this, I exploit a linear complementarity relationship between λ_i^p and μ_i^p.

1. If $\lambda_i^p > 0$, then $e_i^p > 0$ and $\mu_i^p = 0$.
2. If $\mu_i^p > 0$, then $e_i^p = 0$ and $\lambda_i^p = 0$.

Rearrange the stationarity condition for e_i^p:

$$\mu_i^p = (c_i - r_i)\lambda_i^p - \sum_{j \neq i} g_{ij}\lambda_j^p + r_i\Lambda_i - \frac{\partial V}{\partial \mathcal{E}^p} \implies \boldsymbol{\mu}^p = \mathbf{M}\boldsymbol{\lambda}^p + \mathbf{R}\boldsymbol{\Lambda} - \left(\frac{\partial V}{\partial \mathcal{E}^p}\right)\mathbf{1}_n$$

where $\mathbf{M} := \mathbf{C} - \mathbf{R} - \mathbf{G}$ and $\boldsymbol{\Lambda} = [\Lambda_1, ..., \Lambda_n]^\intercal$. For shorthand notation, I write $\mathbf{w}^p = \mathbf{R}\boldsymbol{\Lambda} - \left(\frac{\partial V}{\partial \mathcal{E}^p}\right)\mathbf{1}_n$.

So far, for each project p, we have effectively formulated a linear complementarity problem in the Lagrange multipliers $\boldsymbol{\mu}^p$ and $\boldsymbol{\lambda}^p$, which I label $\texttt{LCP}(\mathbf{M}, \mathbf{w}^p)$. Since $\mathbf{H}$ is positive definite by Assumption 1, $\mathbf{M}$ is an M-matrix. The following are two results established by linear complementarity theory.

1. Because $\mathbf{M}$ is positive definite, $\texttt{LCP}(\mathbf{M}, \mathbf{w}^p)$ has a unique solution for each right-hand vector $\mathbf{w}^p$ (Theorem 3.1.6 of Cottle et al. (2009)).

2. Because $\mathbf{M}$ is a Z-matrix, the unique solution $\boldsymbol{\lambda}^*(\mathbf{w}^p)$ to $\texttt{LCP}(\mathbf{M}, \mathbf{w}^p)$ is the least element of the feasible set (Theorem 3.11.6 of Cottle et al. (2009)).

These two facts help us establish the following monotonicity result.

Lemma 2. *If $\mathbf{w} \leq \mathbf{w}'$ (component-wise), then $\boldsymbol{\lambda}^*(\mathbf{w}) \geq \boldsymbol{\lambda}^*(\mathbf{w}')$ and $\boldsymbol{\mu}^*(\mathbf{w}) \leq \boldsymbol{\mu}^*(\mathbf{w}')$.*

Proof. Let $\mathbf{w} \leq \mathbf{w}'$ (component-wise). Then $\mathbf{M}\boldsymbol{\lambda} + \mathbf{w}' \geq \mathbf{M}\boldsymbol{\lambda} + \mathbf{w}$. This implies that $\{\boldsymbol{\lambda} \mid \mathbf{M}\boldsymbol{\lambda} + \mathbf{w} \geq \mathbf{0}\} \subseteq \{\boldsymbol{\lambda} \mid \mathbf{M}\boldsymbol{\lambda} + \mathbf{w}' \geq \mathbf{0}\}$. By the least-element property, the solution $\boldsymbol{\lambda}^*$ must satisfy:

$$\boldsymbol{\lambda}^*(\mathbf{w}') = \min_{\boldsymbol{\lambda}}\{\boldsymbol{\lambda} \mid \mathbf{M}\boldsymbol{\lambda} + \mathbf{w}' \geq \mathbf{0}\} \leq \min_{\boldsymbol{\lambda}}\{\boldsymbol{\lambda} \mid \mathbf{M}\boldsymbol{\lambda} + \mathbf{w} \geq \mathbf{0}\} = \boldsymbol{\lambda}^*(\mathbf{w})$$

The above establishes reverse monotonicity of $\boldsymbol{\lambda}^*$. Now I show the monotonicity of $\boldsymbol{\mu}^*$. Fix some index i.

<u>Case #1:</u> $\boldsymbol{\lambda}^*(\mathbf{w})_i > 0$.

By complementarity, $\boldsymbol{\mu}^*(\mathbf{w})_i = 0$. Since $\boldsymbol{\mu}^*(\mathbf{w}') \geq \mathbf{0} \implies \boldsymbol{\mu}^*(\mathbf{w}')_i \geq \boldsymbol{\mu}^*(\mathbf{w})_i$.

<u>Case #2:</u> $\boldsymbol{\lambda}^*(\mathbf{w})_i = 0$. Because $\boldsymbol{\lambda}^*(\mathbf{w}') \leq \boldsymbol{\lambda}^*(\mathbf{w})$, we have that $\boldsymbol{\lambda}^*(\mathbf{w}')_i = 0$ as well. Then we can write:

$$\boldsymbol{\mu}^*(\mathbf{w})_i - \boldsymbol{\mu}^*(\mathbf{w}')_i = \big(\mathbf{M}(\boldsymbol{\lambda}^*(\mathbf{w}) - \boldsymbol{\lambda}^*(\mathbf{w}'))\big)_i + w_i - w'_i$$

Now we use the fact that $\mathbf{M}$ is a Z-matrix. Since $\boldsymbol{\lambda}^*(\mathbf{w})_i - \boldsymbol{\lambda}^*(\mathbf{w}')_i = 0$ and $\boldsymbol{\lambda}^*(\mathbf{w})_j - \boldsymbol{\lambda}^*(\mathbf{w}')_j \geq 0$ for all $j \neq i$,

$$\big(\mathbf{M}(\boldsymbol{\lambda}^*(\mathbf{w}) - \boldsymbol{\lambda}^*(\mathbf{w}'))\big)_i = \sum_{j \neq i} \mathbf{M}_{ij}(\boldsymbol{\lambda}^*(\mathbf{w})_j - \boldsymbol{\lambda}^*(\mathbf{w}')_j) \leq 0$$

Since $w'_i \geq w_i$, we can conclude that $\boldsymbol{\mu}^*(\mathbf{w})_i - \boldsymbol{\mu}^*(\mathbf{w}')_i \leq 0$, completing the proof. □

Returning to the main proof, assume that projects p and q are such that $\frac{\partial V}{\partial \mathcal{E}^p} > \frac{\partial V}{\partial \mathcal{E}^q}$.

$$\mathbf{w}^p = \mathbf{R}\boldsymbol{\Lambda} - \left(\frac{\partial V}{\partial \mathcal{E}^p}\right)\mathbf{1}_n \leq \mathbf{R}\boldsymbol{\Lambda} - \left(\frac{\partial V}{\partial \mathcal{E}^q}\right)\mathbf{1}_n = \mathbf{w}^q$$

Applying the monotonicity properties of $\boldsymbol{\lambda}^*$ gives $\boldsymbol{\lambda}^{p*} \geq \boldsymbol{\lambda}^{q*}$ and $\boldsymbol{\mu}^{p*} \leq \boldsymbol{\mu}^{q*}$. Therefore, there exist cutoffs project cutoffs σ_i and γ_i such that

$$p \geq \sigma_i \iff \mu_i^p = 0 \iff i \in A^p \qquad p \geq \gamma_i \iff \lambda_i^p > 0 \iff i \in \mathcal{F}^p$$

$\sigma_i \geq \gamma_i$, otherwise there exists project p such that $i \in \mathcal{F}^p$ but $i \notin A^p$, which is ruled out by Assumption 3. For the last statement, order projects by the magnitude of $\frac{\partial V}{\partial \mathcal{E}^p}$ in the optimum such that

$$\frac{\partial V}{\partial \mathcal{E}^1} \geq \cdots \geq \frac{\partial V}{\partial \mathcal{E}^m}$$

If worker i is fully utilized at project q i.e. $\lambda_i^q > 0$, then $\lambda_i^p > 0$. This implies that the set of workers fully utilized at project q is a subset of those fully utilized at project p. We arrive at the desired nested structure

$$\mathcal{F}^1 \supseteq \mathcal{F}^2 \supseteq \cdots \supseteq \mathcal{F}^m.$$

where $\mathcal{F}^p = \{i \mid \lambda_i^p > 0\}$.

If $e_i^{q*} > 0$, then $\mu_i^q = 0$ and so $\mu_i^p = 0$. This implies the workers assigned positive effort at project q are also assigned positive effort at project p i.e. $A^q \subseteq A^p$ for all $p < q$ where $A^q = \{i \mid e_i^{p*} > 0\}$, ignoring the knife-edge case where $e_i^p = 0$ and $\mu_i^p = 0$. We arrive at the desired nested structure

$$A^1 \supseteq A^2 \supseteq \cdots \supseteq A^m.$$

□

A.5 Proof of Proposition 3

Assume that $V^p = V^q$. Rearrange the stationarity condition, which reads

$$V^p + \textstyle\sum_{j \neq i} g_{ij} \lambda_j^p + \mu_i^p = (c_i - r_i)\lambda_i^p + r_i \Lambda_i$$

where λ_i^p and μ_i^p are the multipliers on the (i, p) incentive and nonnegativity constraints and $\Lambda_i := \sum_{p=1}^m \lambda_i^p$. Subtract the stationarity condition for (i, q) from (i, p).

The multipliers satisfy the following equality

$$\textstyle\sum_{j\neq i} g_{ij}(\lambda_j^p - \lambda_j^q) + (\mu_i^p - \mu_i^q) = (c_i - r_i)(\lambda_j^p - \lambda_j^q).$$

Stack the equations for each worker

$$\boldsymbol{\mu}^p - \boldsymbol{\mu}^q = \mathbf{M}(\boldsymbol{\lambda}^p - \boldsymbol{\lambda}^q).$$

Suppose the two projects are staffed differently so that $\mathbf{d} := \boldsymbol{\lambda}^p - \boldsymbol{\lambda}^q \neq \mathbf{0}$. Because $\mathbf{M}$ is positive definite by Assumption 1, $\mathbf{d}^\intercal\mathbf{M}\mathbf{d} > \mathbf{0}$ while the left hand side has $\mathbf{d}^\intercal(\boldsymbol{\mu}^p - \boldsymbol{\mu}^q) = -(\boldsymbol{\lambda}^p\mu^q - \boldsymbol{\lambda}^q\mu^p) \leq 0$ which is a contradiction. Thus it must be that $\boldsymbol{\lambda}^p = \boldsymbol{\lambda}^q$ and $\boldsymbol{\mu}^p = \boldsymbol{\mu}^q$.

For the second statement, fix a worker i underutilized at projects p and q and without loss of generality let $V^p \geq V^q$. Underutilization requires $\mu_i^p = 0$ and $\lambda_i^p = 0$, and likewise at q. Row i of the stationarity condition therefore loses its own-multiplier term $(c_i - r_i)\lambda_i^p$ and collapses to

$$V^p = r_i\Lambda_i - \sum_{j\neq i} g_{ij}\lambda_j^p$$

and identically at q. Differencing the two, the term $r_i\Lambda_i$ cancels because it carries no project index:

$$V^p - V^q = \sum_{j\neq i} g_{ij}\big(\lambda_j^q - \lambda_j^p\big).$$

Since $V^p \geq V^q$, we have $w^p \leq w^q$ componentwise, so Lemma 2 gives $\lambda^p \geq \lambda^q$. As $g_{ij} \geq 0$, the right-hand side is nonpositive, so $V^p \leq V^q$ and therefore $V^p = V^q$. Because p and q were arbitrary projects at which worker i is underutilized, $V^{\gamma_i+1} = \cdots = V^{\sigma_i}$. □

A.6 Proof of Proposition 4

The KKT condition of the principal's problem, which is necessary and sufficient for optimality as all constraints are linear in efforts and the objective is concave, in vector form, is:

$$\nabla_{\mathbf{e}} V(\mathcal{E}^1, ..., \mathcal{E}^m) + ((\mathbf{G} \otimes \mathbf{I}_m) - (\mathbf{C} \otimes \mathbf{I}_m + \mathbf{R} \otimes (\mathbf{1}_m \mathbf{1}_m^\intercal - \mathbf{I}_m)))\boldsymbol{\lambda} + \boldsymbol{\mu} = \mathbf{0}$$

where $\boldsymbol{\lambda} \geq \mathbf{0}$ and $\boldsymbol{\mu} \geq \mathbf{0}$ are the vectors of Lagrange multipliers for the incentive constraints and nonnegativity conditions. This is equivalent to $\nabla_{\mathbf{e}} V(\mathcal{E}^1, ..., \mathcal{E}^m) = \mathbf{H}\boldsymbol{\lambda} - \boldsymbol{\mu}$. If $(i, p) \in A^p$, then $\frac{\partial V}{\partial \mathcal{E}^p} = (\mathbf{H}\boldsymbol{\lambda})_i^p$. If $(i, p) \notin A^p$, then $\frac{\partial V}{\partial \mathcal{E}^p} \leq (\mathbf{H}\boldsymbol{\lambda})_i^p$. If every worker is assigned and fully utilized at every project, then $\boldsymbol{\mu} = \mathbf{0}$, and $\nabla_{\mathbf{e}} V(\mathcal{E}^1, ..., \mathcal{E}^m) = \mathbf{H}\boldsymbol{\lambda}$, which means $\nabla_{\mathbf{e}} V$ lies in the convex cone generated by the columns of the matrix $\mathbf{H}$. □

A.7 Proof of Proposition 5

I first show that there always exists a worker-symmetric optimal project assignment. Note that the principal's objective is strictly concave over $(\mathcal{E}^1, ..., \mathcal{E}^m)$. Since the feasible set is convex, there exists a unique maximizer in total efforts per project $(\mathcal{E}^{1*}, ..., \mathcal{E}^{m*})$. Now, take any optimal project assignment $\{\{\hat{e}_i^p\}_{p=1}^m\}_{i=1}^n$ and consider the average assignment:

$$\bar{e}_i^p := \bar{e}^{p*} = \frac{1}{n} \sum_{j=1}^n \hat{e}_j^p$$

The total efforts at each project remain unchanged, and so $\{\{\bar{e}_i^p\}_{p=1}^m\}_{i=1}^n$ also achieves the principal's maximum. To show it is feasible, note that since $\{\{\hat{e}_i^p\}_{p=1}^m\}_{i=1}^n$ is feasible, then:

$$\alpha + g \sum_{j \neq i} \hat{e}_j^p \geq c\hat{e}_i^p + r \sum_{q \neq p} \hat{e}_i^q \implies n\alpha + g \sum_{j \neq i} \sum_{\ell=1}^n \hat{e}_\ell^p \geq c \sum_{j=1}^n \hat{e}_j^p + r \sum_{q \neq p} \sum_{j=1}^n \hat{e}_i^q$$

Now we need to show the incentive constraints for $\{\{\bar{e}_i^p\}_{p=1}^m\}_{i=1}^n$ also holds. Take any incentive constraint.

$$
\begin{aligned}
\alpha + g\sum_{j\neq i}\bar{e}_j^p - c\bar{e}_i^p - r\sum_{q\neq p}\bar{e}_i^q &= \alpha + g\sum_{j\neq i}\left(\frac{1}{n}\sum_{\ell=1}^n \hat{e}_\ell^p\right) - c\left(\frac{1}{n}\sum_{j=1}^n \hat{e}_j^p\right) - r\sum_{q\neq p}\left(\frac{1}{n}\sum_{j=1}^n \hat{e}_j^q\right) \\
&= \alpha + g\left(\frac{n-1}{n}\sum_{\ell=1}^n \hat{e}_\ell^p\right) - c\left(\frac{1}{n}\sum_{j=1}^n \hat{e}_j^p\right) - r\sum_{q\neq p}\left(\frac{1}{n}\sum_{j=1}^n \hat{e}_j^q\right) \\
&= n\alpha + g\left((n-1)\sum_{\ell=1}^n \hat{e}_\ell^p\right) - c\left(\sum_{j=1}^n \hat{e}_j^p\right) - r\sum_{q\neq p}\left(\sum_{j=1}^n \hat{e}_j^q\right) \\
&= n\alpha + g\sum_{j\neq i}\sum_{\ell=1}^n \hat{e}_\ell^p - c\sum_{j=1}^n \hat{e}_j^p - r\sum_{q\neq p}\sum_{j=1}^n \hat{e}_j^q \\
&\geq 0
\end{aligned}
$$

(1) In the worker-symmetric optimal project assignment, every worker plays the same optimal effort at every project. Because the Cobb-Douglas objective satisfies the Inada conditions, every worker must play positive effort at every project.

(2) The worker-symmetric optimal project assignment solves the symmetric problem:

$$
\begin{aligned}
&\max_{\{e^p\}_{p=1}^m} \sum_{p=1}^m \beta^{(p)} \ln(e^p) \\
&\text{s.t. } \alpha + g(n-1)e^p \geq ce^p + r\sum_{q\neq p} e^q
\end{aligned}
$$

A unique maximizer $\{e^{p*}\}_{p=1}^m$ exists, and the KKT conditions are both necessary and sufficient. Consider project s and rewrite the incentive constraint in terms of the total effort in the entire system:

$$
\alpha + g(n-1)e^s \geq ce^s + r\sum_{p\neq s} e^p \implies \alpha + (g(n-1) - c + r)e^s \geq r\sum_{p=1}^m e^p
$$

For any projects s at which the incentive constraint binds, we have:

$$
\alpha + (g(n-1) - c + r)e^{s*} = r\sum_{p=1}^m e^{p*} \implies e^{s*} = \frac{\alpha - r\sum_{p=1}^m e^{p*}}{c - r - g(n-1)}
$$

For project t with a slack incentive constraint, we have

$$\alpha + (g(n-1) - c + r)e^{t*} > \sum_{q=1}^{m} e^{q*} \implies e^{t*} < \frac{\alpha - r\sum_{q=1}^{m} e^{q*}}{c - r - g(n-1)}$$

This means $e^{t*} < e^{s*}$ for any project t and s with slack and binding incentive constraints, respectively.

All projects with binding incentive constraints have the same optimal effort, which depends on the total effort in the entire system. The first-order condition for any project s with a binding incentive constraint is

$$\frac{\beta^{(s)}}{e^{s*}} = \lambda^{s*}(c - g(n-1) - r) + r\sum_{p=1}^{m} \lambda^{p*} \geq r\sum_{p=1}^{m} \lambda^{p*}$$

For project t with a slack incentive constraint ($\lambda^{t*} = 0$), its first-order condition is:

$$\frac{\beta^{(t)}}{e^{t*}} + \lambda^{t*}(g(n-1) - c + r) - r\sum_{q=1}^{m} \lambda^{q*} = 0 \implies \frac{\beta^{(t)}}{e^{t*}} = r\sum_{p=1}^{m} \lambda^{p*}$$

Linking both inequalities together yields:

$$\beta^{(s)} \geq e^{s*}\left(r\sum_{p=1}^{m} \lambda^{q*}\right) > e^{t*}\left(r\sum_{q=1}^{m} \lambda^{q*}\right) = \beta^{(t)}$$

Every project with a binding incentive constraint has a strictly larger β-weight than every project with a slack incentive constraint. Since projects are ordered $\beta^{(1)} > \cdots > \beta^{(m)}$, that means there exists a project index ζ^* that for all projects $s \leq \zeta^*$, $\beta^{(s)} \geq \beta^{(\zeta*)}$ with a binding incentive constraint and for all projects $t > \zeta^*$, $\beta^{(t)} < \beta^{(\zeta*)}$ with a slack incentive constraint. The formula for ζ^* is derived in **(4)**.

(3) I first compute the optimal total effort in the entire system, which is a sufficient statistic for efforts at projects with a binding incentive constraint. All projects with binding incentive constraints have the same optimal effort. The following substitutes in the first-order condition for projects with slack incentive constraints.

$$\sum_{p=1}^{m} e^{p*} = \sum_{p=1}^{\zeta^*} e^{p*} + \sum_{t=\zeta^*+1}^{m} e^{t*} = \zeta^* e^{s*} + \left(\frac{1}{r\sum_{p=1}^{m} \lambda^{p*}}\right) \sum_{t=\zeta^*+1}^{m} \beta^{(t)}$$

From the binding incentive constraint on project s, we obtain:

$$\alpha + (g(n-1) - c + r)e^{s*} = r\sum_{p=1}^{m} e^{p*} \implies \alpha = (c - r - g(n-1) + r\zeta^*)e^{s*} + \left(\frac{1}{\sum_{p=1}^{m}\lambda^{p*}}\right)\sum_{t=q^*+1}^{m}\beta^{(t)}$$

Next, we determine a relationship between the multipliers $\sum_{p=1}^{m}\lambda^{p*}$ and the relative project weights β. Summing the first-order conditions for all projects s with binding incentive constraints gives:

$$\frac{\beta^{(s)}}{e^{s*}} = (c - r - g(n-1))\lambda^{s*} + r\sum_{p=1}^{m}\lambda^{p*} \implies \sum_{p=1}^{\zeta^*}\frac{\beta^{(p)}}{e^{p*}} = (c - r - g(n-1))\sum_{p=1}^{\zeta^*}\lambda^{p*} + \zeta^* r\sum_{p=1}^{m}\lambda^{p*}$$

Note that $\sum_{p=1}^{\zeta^*}\lambda^{p*} = \sum_{p=1}^{m}\lambda^{p*}$ because $\lambda^{t*} = 0$ for all $t \geq \zeta^*$. Together with the fact that all efforts at projects with binding incentive constraints are identical, we can write:

$$\frac{1}{e^{s*}}\sum_{p=1}^{\zeta^*}\beta^{(p)} = (c - r - g(n-1)) + \zeta^* r)\sum_{p=1}^{m}\lambda^{p*} \implies e^{s*}(c - r - g(n-1)) + \zeta^* r) = \frac{\sum_{p=1}^{\zeta^*}\beta^{(p)}}{\sum_{p=1}^{m}\lambda^{p*}}$$

Finally, substitute this into the equation for α:

$$\alpha = \frac{\sum_{p=1}^{\zeta^*}\beta^{(p)}}{\sum_{p=1}^{m}\lambda^{p*}} + \left(\frac{1}{\sum_{p=1}^{m}\lambda^{p*}}\right)\sum_{t=\zeta^*+1}^{m}\beta^{(t)} \implies \sum_{p=1}^{m}\lambda^{p*} = \frac{\sum_{p=1}^{m}\beta^{(p)}}{\alpha}$$

Now we can directly compute e^{s*} and e^{t*}:

$$e^{s*} = \frac{\alpha - \left(\frac{1}{\sum_{p=1}^{m}\lambda^{p*}}\right)\sum_{t=\zeta^*+1}^{m}\beta^{(t)}}{c - r - g(n-1) + r\zeta^*} \implies e^{s*} = \frac{\alpha}{c - r - g(n-1) + \zeta^* r}\left(\frac{\sum_{p=1}^{\zeta^*}\beta^{(p)}}{\sum_{p=1}^{m}\beta^{(p)}}\right)$$

$$e^{t*} = \frac{\beta^{(t)}}{r\sum_{p=1}^{m}\lambda^{p*}} = \frac{\alpha}{r}\left(\frac{\beta^{(s)}}{\sum_{p=1}^{m}\beta^{(p)}}\right)$$

For project s with a binding incentive constraint, the first-order condition inequality with the closed form for e^{s*} gives:

$$\frac{\beta^{(s)}}{e^{s*}} \geq r \sum_{p=1}^{m} \lambda^{p*} \implies \beta^{(s)} \left(\frac{c - r - g(n-1) + \zeta^* r}{\alpha} \right) \left(\frac{\sum_{p=1}^{m} \beta^{(p)}}{\sum_{p=1}^{\zeta^*} \beta^{(p)}} \right) \geq r \left(\frac{\sum_{p=1}^{m} \beta^{(p)}}{\alpha} \right)$$

The right-hand inequality simplifies to

$$\frac{\beta^{(s)}}{r} \geq \frac{\sum_{p=1}^{\zeta^*} \beta^{(p)}}{c - r - g(n-1) + \zeta^* r}$$

Since this inequality holds for all $s \leq \zeta^*$, it must be that

$$\frac{\beta^{\zeta^*}}{r} \geq \frac{\sum_{p=1}^{\zeta^*} \beta^{(p)}}{c - r - g(n-1) + \zeta^* r}$$

Note that project $\zeta^* + 1$ must have a slack incentive constraint. We can substitute the formulas for e^{q*} and e^{q*+1} into the inequality $e^{\zeta^*+1} < e^{q*}$:

$$\frac{\alpha}{r} \left(\frac{\beta^{(\zeta^*+1)}}{\sum_{p=1}^{m} \beta^{(p)}} \right) < \frac{\alpha}{c - r - g(n-1) + \zeta^* r} \left(\frac{\sum_{p=1}^{\zeta^*} \beta^{(p)}}{\sum_{p=1}^{m} \beta^{(p)}} \right) \implies \frac{\beta^{\zeta^*+1}}{r} < \frac{\sum_{p=1}^{\zeta^*} \beta^{(p)}}{c - r - g(n-1) + \zeta^* r}$$

Thus ζ^* is a natural number that satisfies:

$$\frac{\beta^{(\zeta^*+1)}}{r} < \frac{\sum_{p=1}^{\zeta^*} \beta^{(p)}}{c - r - g(n-1) + \zeta^* r} \leq \frac{\beta^{(\zeta^*)}}{r}$$

If there was another natural number $\hat{q} \neq \zeta^*$ that satisfies the above thresholds, then $\hat{q}$ induce $\hat{e}^s \neq e^{s*}$ and $\hat{e}^t \neq e^{t*}$ for all projects with binding and slack incentive constraints, respectively. However, this contradicts the uniqueness of the optimal e^*. From the above, we can immediately see that ζ^* is the maximum project index that satisfies

$$\frac{\sum_{p=1}^{\zeta} \beta^{(\zeta)}}{c - r - g(n-1) + \zeta r} \leq \frac{\beta^{(\zeta)}}{r} \iff \frac{\sum_{p=1}^{\zeta} \beta^{(p)}}{\beta^{(\zeta)}} - \zeta \leq \frac{c - r - g(n-1)}{r}.$$

I show that ζ^* is non-increasing in r using the established bounds about ζ^*. From the right inequality, we have the equivalent statement

$$\frac{\sum_{p=1}^{\zeta^*}\beta^{(p)}}{c-r-g(n-1)+\zeta^* r} \leq \frac{\beta^{(\zeta^*)}}{r} \iff r\left(\sum_{p=1}^{\zeta^*}\beta^{(p)} - (\zeta^*-1)\beta^{(\zeta^*)}\right) \leq \beta^{(\zeta^*)}(c-g(n-1))$$

Because we ordered $\beta^{(1)} > \cdots > \beta^{(m)}$,

$$\sum_{p=1}^{\zeta^*}\beta^{(p)} - \zeta^*\beta^{(\zeta^*)} > 0 \implies \sum_{p=1}^{\zeta^*}\beta^{(p)} - (\zeta^*-1)\beta^{(\zeta^*)} > 0$$

We derive the following upper bound on r in terms of ζ^*:

$$r \leq \frac{\beta^{(\zeta^*)}(c-g(n-1))}{\sum_{p=1}^{\zeta^*}\beta^{(p)} - (\zeta^*-1)\beta^{(\zeta^*)}} := U_{\zeta^*}$$

From the left inequality that ζ^* satisfies in (4a), we have the equivalent statement:

$$\frac{\beta^{(\zeta^*+1)}}{r} < \frac{\sum_{p=1}^{\zeta^*}\beta^{(p)}}{c-r-g(n-1)+\zeta^* r} \iff r\left(\sum_{p=1}^{\zeta^*}\beta^{(p)} - (\zeta^*-1)\beta^{(\zeta^*+1)}\right) > \beta^{(\zeta^*+1)}(c-g(n-1))$$

This establishes a lower bound on r in terms of ζ^*:

$$r > \frac{\beta^{(\zeta^*+1)}(c-g(n-1))}{\sum_{p=1}^{\zeta^*}\beta^{(p)} - (\zeta^*-1)\beta^{(\zeta^*+1)}} := L_{\zeta^*}$$

Note that $U_{q+1} = L_q$ and U_q is strictly decreasing in q. I show this by computing the discrete difference

$$U_{q+1} - U_q = \frac{(\beta^{(q+1)} - \beta^{(q)})(c-g(n-1))\sum_{p=1}^{q}\beta^{(p)}}{(\sum_{p=1}^{q}\beta^{(p)} - (q-1)\beta^{(q)})(\sum_{p=1}^{q}\beta^{(p)} - (q-1)\beta^{(q+1)})} < 0$$

Thus we have partitioned the r-space into m disjoint intervals:

$$[L_m, U_m) \cup [L_{m-1}, U_{m-1}) \cup \cdots \cup [L_1, U_1)$$

where $L_m = 0$ and $U_1 = c - g(n-1)$. For any r that satisfies the strict diagonal dominance condition i.e. $r \in [0, c-g(n-1))$, there exists an unique ζ^* such that

$L_{\zeta^*} \leq r < U_{\zeta^*}$. Thus ζ^* is a non-increasing step function of r given by:

$$\zeta^*(r) = \max\{q \mid r < U_q\}$$

The bounds U_q are increasing in c and decreasing in g and n. Therefore, ζ^* is a non-decreasing step function in c and a non-increasing step function in g and n. Because U_q is independent of α, then ζ^* is also independent of α. □

A.8 Proof of Theorem 2

I begin by proving the ordering of the project marginal values.

Lemma 3. *In any optimum,* $\frac{\beta^1}{\mathcal{E}^{1*}} \geq \cdots \geq \frac{\beta^m}{\mathcal{E}^{m*}}$.

Proof. I perform a column swap on the dual problem to reach a contradiction. Writing out the Lagrangian

$$\mathcal{L} = \sum_{p=1}^{m} \beta^{(p)} \ln(\mathcal{E}^p) + \sum_{p=1}^{m} \sum_{i=1}^{n} \lambda_i^p (\alpha + g \sum_{j \neq i} e_j^p - c e_i^p - r_i \sum_{q \neq p} e_i^q)$$

Define the multiplier sums $\Lambda_i := \sum_{p=1}^{m} \lambda_i^p$ and $\Lambda^p := \sum_{i=1}^{n} \lambda_i^p$. I rewrite the Lagrangian as

$$\begin{aligned}
\mathcal{L} &= \sum_{p=1}^{m} \beta^{(p)} \ln(\mathcal{E}^p) + \sum_{p=1}^{m} \sum_{i=1}^{n} e_i^p ((r_i - c - g)\lambda_i^p + g\Lambda^p - r_i \Lambda_i) + \sum_{p=1}^{m} \sum_{i=1}^{n} \alpha \lambda_i^p \\
&= \sum_{p=1}^{m} \beta^{(p)} \ln(\mathcal{E}^p) + \sum_{p=1}^{m} \mathcal{E}^p g \Lambda^p + \sum_{p=1}^{m} \sum_{i=1}^{n} e_i^p ((r_i - c - g)\lambda_i^p - r_i \Lambda_i) + \sum_{p=1}^{m} \sum_{i=1}^{n} \alpha \lambda_i^p
\end{aligned}$$

To form the dual, I maximize the Lagrangian over all $\mathbf{e}$ given a fixed $\boldsymbol{\lambda} \geq \mathbf{0}$. The maximization problem is separable across projects; the following is it for project p:

$$\max_{\{e_i^p\}_{i=1}^n} \beta^{(p)} \ln(\mathcal{E}^p) + \mathcal{E}^p g \Lambda^p + \sum_{i=1}^{n} e_i^p ((r_i - c - g)\lambda_i^p - r_i \Lambda_i)$$

Observe that $((r_i - c - g)\lambda_i^p - r_i \Lambda_i) \leq 0$ for all i. Then, fixing the total effort $\mathcal{E}^p$, the objective is maximized by setting $e_i^p = \mathcal{E}^p$ for worker i with the maximum

$((r_i - c - g)\lambda_i^p - r_i\Lambda_i)$ term. Therefore, we can rewrite the maximization problem

$$\max_{\mathcal{E}^p} \beta^{(p)} \ln(\mathcal{E}^p) + \mathcal{E}^p g\Lambda^p + \mathcal{E}^p \max_i \{(r_i - c - g)\lambda_i^p - r_i\Lambda_i\}$$

The unique maximizer is

$$\mathcal{E}^{p*} = \frac{\beta^{(p)}}{-\max_i\{(r_i - c - g)\lambda_i^p - r_i\Lambda_i\} - g\Lambda^p}$$

With some cancellations, we can form the dual function

$$D(\boldsymbol{\lambda}) = \sum_{p=1}^{m} \left(\beta^{(p)} \ln(\mathcal{E}^{p*}) - \beta^{(p)}\right) + \sum_{p=1}^{m}\sum_{i=1}^{n} \alpha\lambda_i^p$$

Slater's condition for the primal holds, implying strong duality. Now assume by contradiction that $\boldsymbol{\lambda}^*$ minimizes the dual and that there exists $p < q$ with $\frac{\beta^p}{\mathcal{E}^{p*}} < \frac{\beta^q}{\mathcal{E}^{q*}}$. Construct a swap $\tilde{\boldsymbol{\lambda}}$ by setting:

$$\tilde{\lambda}_i^p = \lambda_i^{q*} \ \ ; \ \ \tilde{\lambda}_i^q = \lambda_i^{p*} \ \ ; \ \ \tilde{\lambda}_i^s = \lambda_i^{s*} \ \ \forall s \neq p, q$$

Observe that $\tilde{\Lambda}_i = \Lambda_i^*$ for all workers i, however the project multiplier sums for p and q are swapped i.e. $\tilde{\Lambda}^p = \Lambda^{q*}$ and $\tilde{\Lambda}^q = \Lambda^{p*}$. This means that for project p,

$$\max_i \{(r_i - c - g)\tilde{\lambda}_i^p - r_i\tilde{\Lambda}_i\} = \max_i \{(r_i - c - g)\lambda_i^{q*} - r_i\Lambda_i^*\}$$

The same swap occurs for project q. Now compute:

$$\begin{aligned} D(\tilde{\boldsymbol{\lambda}}) - D(\boldsymbol{\lambda}^*) = &-\beta^{(p)} \ln(-\max_i \{(r_i - c - g)\lambda_i^{q*} - r_i\Lambda_i^*\} - g\Lambda^{q*}) \\ &- \beta^{(q)} \ln(-\max_i \{(r_i - c - g)\lambda_i^{p*} - r_i\Lambda_i^*\} - g\Lambda^{p*}) + \alpha \left(\sum_{i=1}^{n} \lambda_i^{p*} + \sum_{i=1}^{n} \lambda_i^{q*}\right) \\ &+ \beta^{(p)} \ln(-\max_i \{(r_i - c - g)\lambda_i^{p*} - r_i\Lambda_i^*\} - g\Lambda^{p*}) \\ &+ \beta^{(q)} \ln(-\max_i \{(r_i - c - g)\lambda_i^{q*} - r_i\Lambda_i^*\} - g\Lambda^{q*}) - \alpha \left(\sum_{i=1}^{n} \lambda_i^{q*} + \sum_{i=1}^{n} \lambda_i^{q*}\right) \end{aligned}$$

Combining like terms yields

$$\beta^p \ln\left(\frac{-\max_i\{(r_i - c - g)\lambda_i^{p*} - r_i\Lambda_i^*\} - g\Lambda^{p*}}{-\max_i\{(r_i - c - g)\lambda_i^{q*} - r_i\Lambda_i^*\} - g\Lambda^{q*}}\right) + \beta^q \ln\left(\frac{-\max_i\{(r_i - c - g)\lambda_i^{q*} - r_i\Lambda_i^*\} - g\Lambda^{q*}}{-\max_i\{(r_i - c - g)\lambda_i^{p*} - r_i\Lambda_i^*\} - g\Lambda^{p*}}\right)$$

Recall the unique maximizer $\mathcal{E}^{p*}$ satisfies

$$\frac{\beta^{(p)}}{\mathcal{E}^{p*}} = -\max_i\{(r_i - c - g)\lambda_i^{p*} - r_i\Lambda_i^*\} - g\Lambda^{p*}$$

Substituting back in, we obtain

$$\beta^{(p)} \ln\left(\frac{\beta^{(p)}/\mathcal{E}^{p*}}{\beta^{(q)}/\mathcal{E}^{q*}}\right) + \beta^{(q)} \ln\left(\frac{\beta^{(q)}/\mathcal{E}^{q*}}{\beta^{(p)}/\mathcal{E}^{p*}}\right) = (\beta^{(p)} - \beta^{(q)}) \ln\left(\frac{\beta^{(p)}/\mathcal{E}^{p*}}{\beta^{(q)}/\mathcal{E}^{q*}}\right) < 0$$

The inequality holds because $p < q \implies \beta^{(p)} - \beta^{(q)} > 0$ and our original assumption that $\frac{\beta^{(p)}}{\mathcal{E}^{p*}} < \frac{\beta^{(q)}}{\mathcal{E}^{q*}}$. But this implies $D(\tilde{\boldsymbol{\lambda}}) - D(\boldsymbol{\lambda}^*) < 0$, which contradicts minimality of $\boldsymbol{\lambda}^*$. Thus we must have $\frac{\beta^{(p)}}{\mathcal{E}^{p*}} \geq \frac{\beta^{(q)}}{\mathcal{E}^{q*}}$. $\square$

Since $\frac{\partial V}{\partial \mathcal{E}^p} = \frac{\beta^{(p)}}{\mathcal{E}^p}$, we can apply Theorem 1 to conclude that A^p and $\mathcal{F}^p$ both obey the same project ordering given by the project weights $\beta^{(p)}$. Thus, there exists a worker-specific project index γ_i^* such that $e_i^{p*} > 0$ if and only if $p \leq \gamma_i^*$. To relate γ_i^* to r_i, I explicitly solve for λ_i^p and μ_i^p by the following case analysis. Note that the linear complementarity problem that the multipliers solve is given by

$$\boldsymbol{\mu}^p = \mathbf{M}\boldsymbol{\lambda}^p + \mathbf{R}\boldsymbol{\Lambda} - \left(\frac{\beta^{(p)}}{\mathcal{E}^p}\right)\mathbf{1}_n \ ; \ \boldsymbol{\lambda}^p, \boldsymbol{\mu}^p \geq \mathbf{0} \ ; \ (\boldsymbol{\lambda}^p)^\intercal \boldsymbol{\mu}^p = \mathbf{0}$$

For notation, let $\Lambda^p := \sum_{i=1}^n \lambda_i^p$.

<u>Case #1.1:</u> $e_i^p > 0$.

Then $\mu_i^p = 0$. From the linear complementarity problem, we have

$$0 = \lambda_i^p(c + g - r_i) + r_i\Lambda_i - \frac{\beta^{(p)}}{\mathcal{E}^p} - g\Lambda^p = 0 \implies \lambda_i^p = \frac{\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p - r_i\Lambda_i}{c + g - r_i}$$

Since $\lambda_i^p \geq 0$, it must be that $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p \geq r_i\Lambda_i$.

Case #1.2: $e_i^p = 0$.

Then $\lambda_i^p = 0$ by the generic strict slackness assumption. We have

$$\mu_i^p = r_i\Lambda_i - \frac{\beta^{(p)}}{\mathcal{E}^p} - g\Lambda^p$$

Putting both cases together, we have the following expressions for λ_i^p and μ_i^p:

$$\lambda_i^p = \frac{\max(\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p - r_i\Lambda_i, 0)}{c + g - r_i} \qquad \mu_i^p = \max(r_i\Lambda_i - \frac{\beta^{(p)}}{\mathcal{E}^p} - g\Lambda^p, 0)$$

If $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p > r_i\Lambda_i$, then $\lambda_i^p > 0$. By Assumption 3, $\lambda_i^p > 0$ implies $e_i^p > 0$. If $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p < r_i\Lambda_i$, then $\mu_i^p > 0$, forcing $e_i^p = 0$. Finally, if $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p = r_i\Lambda_i$, then both $\lambda_i^p = \mu_i^p = 0$. This allows us to write $\gamma_i^* = \max\{p \mid \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p \geq r_i\Lambda_i\}$ and also shows that $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p$ is non-increasing in p.

Now I show that $r_i > r_j$ implies $\sigma_i^* \leq \sigma_j^*$. This is equivalent to showing that $r_i\Lambda_i \geq r_j\Lambda_j$. Assume by contradiction that $r_i\Lambda_i < r_j\Lambda_j$, and so $r_j\Lambda_j > 0$ necessarily. There must exist at least one project with $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p > r_j\Lambda_j$; otherwise if $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p \leq r_j\Lambda_j$ for all p, then $\mu_j^p > 0$ for all p, forcing $e_j^p = 0$. By the generic strict slackness assumption, $\lambda_j^p = 0$ for all p, implying $\Lambda_j = 0$. But this contradicts $r_j\Lambda_j > 0$. Therefore, there exists some project p such that $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p > r_j\Lambda_j > r_i\Lambda_i$, implying

$$\lambda_i^p = \frac{\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p - r_i\Lambda_i}{c + g - r_i} > 0\,;\ \lambda_j^p = \frac{\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p - r_j\Lambda_j}{c + g - r_j} > 0$$

Since $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p - r_i\Lambda_i > \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p - r_j\Lambda_j$ and $c + g - r_i < c + g - r_j$, then $\lambda_i^p > \lambda_j^p$ for all p with the property $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p > r_j\Lambda_j$.

Now consider projects q with $\frac{\beta^{(q)}}{\mathcal{E}^q} + g\Lambda^q \in (r_i\Lambda_i, r_j\Lambda_j]$, which have the property

$\lambda_i^q > 0 = \lambda_j^q$. Summing the λ multipliers across all projects.

$$\Lambda_i = \sum_{p \,\mid\, \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p > r_j\Lambda_j} \lambda_i^p + \sum_{p \,\mid\, r_i\Lambda_i < \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p \le r_j\Lambda_j} \lambda_i^p + \underbrace{\sum_{p \,\mid\, \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p \le r_i\Lambda_i} \lambda_i^p}_{=0}$$

$$\Lambda_j = \sum_{p \,\mid\, \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p > r_j\Lambda_j} \lambda_j^p + \underbrace{\sum_{p \,\mid\, r_i\Lambda_i < \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p \le r_j\Lambda_j} \lambda_j^p}_{=0} + \underbrace{\sum_{p \,\mid\, \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p \le r_i\Lambda_i} \lambda_j^p}_{=0}$$

We obtain $\Lambda_i > \Lambda_j \implies r_i\Lambda_i > r_j\Lambda_j$, which is a direct contradiction.

Because $\frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p$ is non-increasing in p, the active set for worker i is $\{p \mid \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p \ge r_i\Lambda_i\}$. Thereby $r_i > r_j \implies r_i\Lambda_i \ge r_j\Lambda_j \implies \{p \mid \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p \ge r_i\Lambda_i\} \subseteq \{p \mid \frac{\beta^{(p)}}{\mathcal{E}^p} + g\Lambda^p \ge r_j\Lambda_j\} \implies \sigma_j^* \le \sigma_i^*$. Similarly, we can conclude that $\gamma_j^* \le \gamma_i^*$. □

A.9 Proof of Theorem 3

The proof that the marginal project valuations for the principal satisfy $\frac{\beta^{(1)}}{\mathcal{E}^{1*}} \ge \cdots \ge \frac{\beta^{(m)}}{\mathcal{E}^{m*}}$ is almost identical to that in Lemma 3 and thus I do not replicate it here. By the argument in Appendix C, we operate under the generic strict slackness assumption that $e_i^p = 0$ implies $\lambda_i^p = 0$ in the optimum. Like before, we have the linear complementarity problem that the multipliers solve, given by

$$\boldsymbol{\mu}^p = \mathbf{M}\boldsymbol{\lambda}^p + \mathbf{R}\boldsymbol{\Lambda} - \left(\frac{\beta^{(p)}}{\mathcal{E}^p}\right)\mathbf{1}_n \; ; \; \boldsymbol{\lambda}^p, \boldsymbol{\mu}^p \ge \mathbf{0} \; ; \; (\boldsymbol{\lambda}^p)^\intercal \boldsymbol{\mu}^p = \mathbf{0}$$

To understand when workers are unassigned from projects due to a decreasing marginal value of projects $\left(\frac{\beta^{(p)}}{\mathcal{E}^p}\right)$, which is a discrete parameter, I turn to its continuous counterpart.

$$\boldsymbol{\mu}^p = \mathbf{M}\boldsymbol{\lambda}^p + \mathbf{R}\boldsymbol{\Lambda} - \rho\mathbf{1}_n \; ; \; \boldsymbol{\lambda}^p, \boldsymbol{\mu}^p \ge \mathbf{0} \; ; \; (\boldsymbol{\lambda}^p)^\intercal \boldsymbol{\mu}^p = \mathbf{0}$$

$\left(\frac{\beta^{(p)}}{\mathcal{E}^p}\right)$ is replaced by $\rho \in \mathbb{R}$. I employ a pivot-type method. Ignoring the nonnegativity of $\boldsymbol{\lambda}^p$, I solve the system assuming that $\boldsymbol{\mu}^p = \mathbf{0}$, i.e., all workers are active on project p. We obtain:

$$\mathbf{M}\boldsymbol{\lambda}^p + \mathbf{R}\boldsymbol{\Lambda} - \sigma\mathbf{1}_n = \mathbf{0} \implies \boldsymbol{\lambda}^p = \mathbf{M}^{-1}(\rho\mathbf{1}_n - \mathbf{R}\boldsymbol{\Lambda})$$

If ρ is sufficiently large, then $\boldsymbol{\lambda}^p > \mathbf{0}$ and all workers are assigned positive effort on project p. As ρ decreases, $\boldsymbol{\lambda}^p$ likewise decreases. The first index i of $\boldsymbol{\lambda}^p$ to reach zero is when

$$\rho = \left(\frac{r(\mathbf{M}^{-1}\boldsymbol{\Lambda})_i}{(\mathbf{M}^{-1}\mathbf{1})_i}\right)$$

and it becomes negative if

$$\rho < \left(\frac{r(\mathbf{M}^{-1}\boldsymbol{\Lambda})_i}{(\mathbf{M}^{-1}\mathbf{1})_i}\right)$$

Thus the index i corresponding to $\arg\max_i \left(\frac{r(\mathbf{M}^{-1}\boldsymbol{\Lambda})_i}{(\mathbf{M}^{-1}\mathbf{1})_i}\right)$ is the first worker to be assigned $\mu_i^p > 0$, else $\lambda_i^p < 0$ which contradicts the latter's nonnegativity. This implies worker i must be unassigned from all projects p with

$$\frac{\beta^{(p)}}{\mathcal{E}^p} < \left(\frac{r(\mathbf{M}^{-1}\boldsymbol{\Lambda})_i}{(\mathbf{M}^{-1}\mathbf{1})_i}\right)$$

On those projects, we can employ the same strategy, but now restrict attention to the first index j of $\boldsymbol{\lambda}^{p,N\setminus\{i\}}$ that reaches zero, which is when

$$\rho = \left(\frac{r((\mathbf{M}^{N\setminus\{i\}})^{-1}\boldsymbol{\Lambda}^{N\setminus\{i\}})_i}{((\mathbf{M}^{N\setminus\{i\}})^{-1}\mathbf{1}^{N\setminus\{i\}})_j}\right)$$

This is precisely because worker i is no longer an active participant on those projects. The same logic extends recursively to all remaining workers. □

A.10 Proof of Proposition 6

Under full utilization and activity, the optimal project assignment has, for each worker i and project p, $e_i^{p*} > 0$, and the incentive constraint binds

$$\alpha_i^p + \sum_{j \neq i} g_{ji} e_j^{p*} = c_i e_i^{p*} + r_i \sum_{q \neq p} e_i^{q*}$$

Therefore, we have

$$\boldsymbol{\alpha}^p = \mathbf{M}\mathbf{e}^p + \mathbf{R}\mathbf{E}\mathbf{1}_m$$

Now sum over all projects to obtain

$$\sum_{p=1}^{m} \boldsymbol{\alpha}^p = \sum_{p=1}^{m} (\mathbf{M}\mathbf{e}^p + \mathbf{R}\mathbf{E}\mathbf{1}_m) \implies \sum_{p=1}^{m} \boldsymbol{\alpha}^p = \mathbf{M}\mathbf{E}\mathbf{1}_m + m\mathbf{R}\mathbf{E}\mathbf{1}_m$$

Therefore the total effort vector satisfies $\mathbf{E}\mathbf{1}_m = \mathbf{P}^{-1} \sum_{p=1}^{m} \boldsymbol{\alpha}^p$. Substituting into the incentive constraint for project p gives

$$\boldsymbol{\alpha}^p = \mathbf{M}\mathbf{e}^p + \mathbf{R}\mathbf{P}^{-1} \sum_{p=1}^{m} \boldsymbol{\alpha}^p \mathbf{1}_m \implies \mathbf{e}^p = \mathbf{M}^{-1}\boldsymbol{\alpha}^p - \mathbf{M}^{-1}\mathbf{R}\mathbf{P}^{-1} \sum_{p=1}^{m} \boldsymbol{\alpha}^p$$

Now we can differentiate

$$\frac{\partial \mathbf{e}^p}{\partial \alpha_j^q} = \mathbf{M}^{-1}\boldsymbol{\epsilon}_j \delta_{pq} - \mathbf{M}^{-1}\mathbf{R}\mathbf{P}^{-1}\boldsymbol{\epsilon}_j$$

Reading off the i-th entry gives the formula in the theorem. All of the other equations are derived similarly.

To show the sign of $\frac{\partial e_i^{p*}}{\partial \alpha_j^q}$, note that its matrix form is given by:

$$\frac{\partial \mathbf{e}^p}{\partial \boldsymbol{\alpha}^q} = \mathbf{M}^{-1}\delta_{pq} - \mathbf{M}^{-1}\mathbf{R}\mathbf{P}^{-1} = \mathbf{M}^{-1}\delta_{pq} - \frac{1}{m}(\mathbf{M}^{-1} - \mathbf{P}^{-1})$$

The second equality hold because $\mathbf{R} = \frac{1}{m}(\mathbf{P} - \mathbf{M})$. Therefore:

$$\frac{\partial \mathbf{e}^p}{\partial \boldsymbol{\alpha}^q} = \begin{cases} \frac{m-1}{m}\mathbf{M}^{-1} + \frac{1}{m}\mathbf{P}^{-1} & p = q \\ -\frac{1}{m}(\mathbf{M}^{-1} - \mathbf{P}^{-1}) & p \neq q \end{cases}$$

Therefore $\frac{\partial e_i^{p*}}{\partial \alpha_j^p} > 0$ for all $j \neq i$ and $\frac{\partial e_i^{p*}}{\partial \alpha_j^q} < 0$ for all $q \neq p$. $\square$

A.11 Proof of Theorem 4

As an abuse of notation, $\mathbf{e}^p$ and $\boldsymbol{\alpha}^p$ both refer to the efforts and preferences of workers restricted to those in A^p. Define $\mathbf{E} = [\mathbf{e}^1 \cdots \mathbf{e}_n]$. In an optimum with full utilization and partial activity, the binding incentive constraint for worker i at project p is

$$\alpha_i^p + \sum_{j \neq i, j \in A^p} g_{ji} e_j^{p*} = c_i e_i^{p*} + r_i \sum_{q \neq p, q \in P_i} e_i^{q*}$$

In vector form, we have

$$\boldsymbol{\alpha}^p = \mathbf{M}^p \mathbf{e}^p + (\mathbf{RE1}_m)^p$$

where $(\mathbf{RE1}_m)^p$ is the vector of total efforts for workers in A^p multiplied by r_i. Solving for $\mathbf{e}^p$ gives

$$\mathbf{e}^p = (\mathbf{M}^p)^{-1} \boldsymbol{\alpha}^p - (\mathbf{M}^p)^{-1} (\mathbf{RE1}_m)^p$$

$\mathbf{M}^p$ is invertible because principal submatrices of positive definite matrices are also positive definite by Sylvester's Criterion. Now I solve for a worker's total effort

$$\mathcal{E}_i = \sum_{p \in P_i} e_i^p = \sum_{p \in P_i} \left(\sum_{j \in A^p} (\mathbf{M}^p)_{ij}^{-1} \alpha_j^p - (\mathbf{M}^p)_{ij}^{-1} r_j \mathcal{E}_j \right)$$

Rearranging, I obtain:

$$\mathcal{E}_i + \sum_{j=1}^{n} \left(r_j \sum_{p \in P_i \cap P_j} (\mathbf{M}^p)_{ij}^{-1} \right) \mathcal{E}_j = \sum_{p \in P_i} \sum_{j \in A^p} (\mathbf{M}^p)_{ij}^{-1} \alpha_j^p$$

Define $\hat{\alpha}_i := \sum_{p \in P_i} \sum_{j \in A^p} (\mathbf{M}^p)_{ij}^{-1} \alpha_j^p$ and $n \times n$ matrix $\mathcal{P}$ with the following entries:

$$\mathcal{P}_{ij} = \delta_{ij} + r_j \sum_{p \in P_i \cap P_j} (\mathbf{M}^p)_{ij}^{-1}$$

Given $\mathcal{P}$, we have the following linear system in workers' total efforts

$$\mathcal{P}(\mathbf{E1}_m) = \hat{\boldsymbol{\alpha}}$$

Because V is strictly concave in total efforts, the optimal total effort is unique and $\mathcal{P}$ is invertible.

$$\mathbf{E1}_m = \mathcal{P}^{-1}\hat{\boldsymbol{\alpha}} \implies \frac{\partial(\mathbf{E1}_m)}{\partial\gamma} = \mathcal{P}^{-1}\boldsymbol{\sigma}^{(\gamma)}$$

where $\boldsymbol{\sigma}^{(\gamma)}$ is defined for each model parameter γ as

$$\sigma_i^{(\alpha_j^q)} = \begin{cases} (\mathbf{M}^q)_{ij}^{-1} & \text{if } i \in A^q \\ 0 & \text{o.w.} \end{cases} \quad ; \quad \sigma_i^{(c_j)} = -\sum_{p \in A_i \cap A_j} (\mathbf{M}^p)_{ij}^{-1} e_j^{p*}$$

$$\sigma_i^{(r_j)} = \sum_{p \in A_i \cap A_j} (\mathbf{M}^p)_{ij}^{-1}(e_j^{p*} - \mathcal{E}_j^*) \ ; \ \sigma_i^{(g_{j\ell})} = \sum_{p \in A_i \cap A_j \cap A_\ell} (\mathbf{M}^p)_{ij}^{-1} e_\ell^{p*} + (\mathbf{M}^p)_{i\ell}^{-1} e_j^{p*}$$

Now we can finally compute:

$$\frac{\partial \mathbf{e}^p}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left((\mathbf{M}^p)^{-1}\boldsymbol{\alpha}^p\right) - \frac{\partial}{\partial \zeta}\left((\mathbf{M}^p)^{-1}(\mathbf{RE1}_m)^p\right)$$

Each of the results follows from direct computation of the above equation and substituting in $\frac{\partial(\mathbf{E1}_m)}{\partial\zeta} = \mathcal{P}^{-1}\boldsymbol{\sigma}^{(\zeta)}$. □

A.12 Proof of Theorem 5

With underutilization, the necessary and sufficient KKT conditions that $(\mathbf{e}, \boldsymbol{\lambda})$ satisfy are:

$$\nabla_{\mathbf{e}} V(\mathbf{e}) - \boldsymbol{\lambda}\mathbf{H} = \mathbf{0}$$

$$\forall (i,p) \in \mathcal{F},\ \alpha_i^p + \sum_{j \in A^p} g_{ji} e_j^p - c_i e_i^p - r_i \sum_{q \neq p, q \in P_i} e_i^q = 0$$

The corresponding Jacobian is

$$\mathbf{J} = \begin{bmatrix} \nabla^2_{\mathbf{e}} V(\mathbf{e}) & \mathbf{H}_{\mathcal{F}} \\ \mathbf{H}_{\mathcal{F}}^{\intercal} & \mathbf{0} \end{bmatrix}$$

Let's show that $\mathbf{J}$ is invertible. Suppose $(\mathbf{x}, \mathbf{y}) \in \ker(\mathbf{J})$, and so

$$\begin{aligned} &\nabla^2_{\mathbf{e}} V(\mathbf{e}^*)\mathbf{x} + (\mathbf{H})_{\mathcal{F}}\mathbf{y} = \mathbf{0} \\ &(\mathbf{H})_{\mathcal{F}}^{\top}\mathbf{x} = \mathbf{0} \end{aligned}$$

Premultiplying the first equation by $\mathbf{x}^{\intercal}$ gives

$$\mathbf{x}^{\intercal}\nabla^2_{\mathbf{e}} V(\mathbf{e}^*)\mathbf{x} + \mathbf{x}^{\intercal}(\mathbf{H})_{\mathcal{F}}\mathbf{y} = 0 \implies \mathbf{x}^{\intercal}\nabla^2_{\mathbf{e}} V(\mathbf{e}^*)\mathbf{x} = \mathbf{0}$$

By assumption that $\nabla^2_{\mathbf{e}} V(\mathbf{e}^*)$ is strictly negative definite on $\ker(\mathbf{H}_{\mathcal{F}}^{\intercal})$, it must be that $\mathbf{x} = 0$. Substituting back into the first equation, we get

$$\mathbf{H}_{\mathcal{F}}\mathbf{y} = 0$$

Since $\mathbf{H}_{\mathcal{F}}$ is invertible, $\mathbf{H}_{\mathcal{F}}$ has full row rank and it must be that $\mathbf{y} = \mathbf{0}$. This concludes the argument that $\mathbf{J}$ is invertible. By the Implicit Function Theorem, we obtain the general formula for how $(\mathbf{e}^*, \boldsymbol{\lambda}^*)$ react to a change in a parameter θ:

$$\begin{pmatrix} \dfrac{\partial \mathbf{e}^*}{\partial \theta} \\ \\ \dfrac{\partial \boldsymbol{\lambda}^*}{\partial \theta} \end{pmatrix} = -\mathbf{J}^{-1} \begin{pmatrix} \nabla^2_{\mathbf{e},\theta} V(\mathbf{e}^*) - \nabla_\theta(\boldsymbol{\lambda}(\mathbf{H})) \\ \nabla_\theta(\boldsymbol{\alpha} - \mathbf{H}_{\mathcal{F}}) \end{pmatrix}$$

Next, I compute the right-hand vectors associated with each model parameter.

$$\mathbf{a}_i^p = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\eta}_i^p \cdot \lambda_i^{p*} \end{pmatrix} \;\; ; \;\; \mathbf{g}_{ji} = \begin{pmatrix} \boldsymbol{\epsilon}_i^p \cdot \lambda_j^{p*} + \boldsymbol{\epsilon}_j^p \cdot \lambda_i^{p*} \\ -\boldsymbol{\eta}_i^p \cdot (e_j^p) - \boldsymbol{\eta}_j^p \cdot (e_i^p) \end{pmatrix}$$

$$\mathbf{c}_i = \begin{pmatrix} -\sum_{p \in P_i} \boldsymbol{\epsilon}_i^p \cdot \lambda_i^{p*} \\ \sum_{p \in P_i} \boldsymbol{\eta}_i^p \cdot e_i^{p*} \end{pmatrix} \;\; ; \;\; \mathbf{r}_i = \begin{pmatrix} -\sum_{p \in P_i} \boldsymbol{\epsilon}_i^p \cdot \sum_{q \neq p} \lambda_i^{q*} \\ \sum_{p \in P_i} \boldsymbol{\eta}_i^p \cdot \sum_{q \neq p} e_i^{q*} \end{pmatrix}$$

To get the final formulas in the theorem, multiply the first row of $\mathbf{J}^{-1}$ by the corresponding right-hand vector for each model parameter. □

A.13 Proof of Proposition 7

Assume that $\mathbf{e}^*$ is an effort profile where worker i is underutilized at p so that $\frac{\partial U_i}{\partial e_i^p} > 0$ at $\mathbf{e}^*$. Since U_i is continuously differentiable, there exists $\epsilon > 0$ where $U_i(\mathbf{e}_i^* + \mathbf{1}_m \cdot \epsilon, \mathbf{e}_{-i}^*) > U_i(\mathbf{e}_i^*, \mathbf{e}_{-i}^*)$. Monotonicity of w_i gives $w_i(\mathbf{e}_i^* + \mathbf{1}_m \cdot \epsilon, \mathbf{e}_{-i}^*) \geq w_i(\mathbf{e}_i^*, \mathbf{e}_{-i}^*)$, and so we have constructed a profitable deviation for worker i. □

B The General MINLP

The principal's general problem is written as a mixed-integer nonlinear program (MINLP).

$$\begin{aligned}
&\max_{\mathbf{e},\mathbf{z}} \; V(\mathcal{E}^1, ..., \mathcal{E}^m) \\
&\text{s.t. } \mathbf{e} \geq \mathbf{0}, \\
&\mathbf{z} \in \{0,1\}^{mn}, \\
&\forall(i,p),\; e_i^p \leq M z_i^p \\
&\forall(i,p),\; \alpha_i^p + \sum_{j\neq i} g_{ij} e_j^p - c_i e_i^p - r_i \sum_{q\neq p} e_i^q \geq -M(1 - z_i^p) \\
&\forall p,\; \sum_{i=1}^{n} e_i^p = \mathcal{E}^p
\end{aligned} \tag{P3}$$

where M is a sufficiently large positive constant.[17] z_i^p is a binary variable that acts as a switch to ensure that if $e_i^p > 0$, then the incentive constraint has to be satisfied. If that is the case, then z_i^p must be 1. Otherwise, if $e_i^p = 0$, then no incentive constraint is needed and z_i^p is set to 0. Switching the incentive constraints turns the principal's problem into a mixed-integer nonlinear programming problem, render-

[17] For example, one can set $M = ||(\mathbf{C} \otimes \mathbf{I}_m - \mathbf{G} \otimes \mathbf{I}_m)^{-1}\boldsymbol{\alpha}||_\infty$. $(\mathbf{C} \otimes \mathbf{I}_m - \mathbf{G} \otimes \mathbf{I}_m)^{-1}\boldsymbol{\alpha}$ gives the vector of maximum efforts workers could play at each project, absent any substitutability across projects. Instead of using a Big-M constant, we can also write the principal's problem with disjunctive constraints: if $z_i^p = 1$, then $\alpha_i^p + \sum_{j\neq i} g_{ij} e_j^p - c_i e_i^p - r_i \sum_{q\neq p} e_i^q \geq 0$; else $e_i^p = 0$.

ing it NP-hard.

To see an explicit example of parameters where the feasible set is nonconvex, take a simple example with one worker i and two projects, p and q. Let the productivity parameters be $\alpha_i^p = 4$, $\alpha_i^q = 1$, $c_i = 0.5$, $r_i = 0.3$. He prefers working on project p four times as much as on project q. The feasible set contains all pair of nonnegative efforts (e_i^p, e_i^q) that satisfy:

$$\begin{aligned} 4 &\geq 0.5e_i^p + 0.3e_i^q \qquad \text{if } e_i^p > 0 \\ 1 &\geq 0.5e_i^q + 0.3e_i^p \qquad \text{if } e_i^q > 0 \end{aligned}$$

Figure 8 illustrates the boundary of the set of implementable efforts. Effort vectors of the form $(e_i^p, 0)$ for $e_i^p \in [3\frac{1}{3}, 4]$ are feasible, as $0.5e_i^p \leq 2$. e_i^q is zero at those efforts, and so his incentive constraint for project q has no bite.

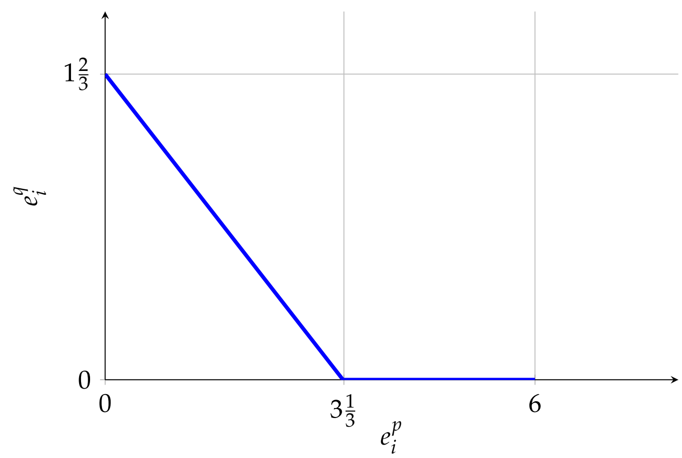


Figure 8: Nonconvex Boundary of Implementable Efforts

The inherent difficulty in analyzing the MINLP comes from the combinatorial nature of the problem. Traditional approaches such as using a greedy algorithm fail because it is neither monotone nor submodular. Increasing the effort cap for a worker at a particular project may cause him to reduce efforts at other projects, leading to an overall decrease in the principal's valuation. Raising effort caps could also cause other workers to increase their efforts on that particular project due to complementarity, increasing the principal's valuation in a convex fashion.

Algorithmic methods, such as outer approximations, serve as efficient ways to compute the solution to the MINLP. Unfortunately, algorithmic approaches provide little in terms of structural and economic insights. Alternatively, absent the sufficient condition on the model parameters in Proposition 2, we can interpret the comparative statics exercises as local analysis treating the binary $\mathbf{z}$ variables in the MINLP as fixed.

C Generic Conditions in the Optimum

Statement (1) is the usual strict complementary slackness condition on λ, which is generic for convex optimization problems that satisfy strict feasibility.

To show that statement (2), $e_i^p = 0$ implies $\lambda_i^p = 0$, is a generic condition, I argue that the set of model parameters that satisfy Assumption 3 with $e_i^p = 0$ and a binding incentive constraint for some worker-project pair (i, p) has measure zero. I label this set as $\mathcal{M}$. Crucially, this *does not* imply the converse i.e. that $\lambda_i^p = 0$ implies $e_i^p = 0$. In the environment with homogeneous workers, underutilizing workers, i.e., $\lambda_i^p = 0$ with $e_i^p > 0$, is a generic feature of the optimal project assignment.

I write $(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r})$ to be an instance of the model parameters. First, note that the planner's objective is strictly concave in total efforts, the feasible set is convex, Slater's condition holds, and the optimal effort totals $\mathcal{E}^p(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r})$ and $\mathcal{E}_i(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r})$ are unique. On an open, dense subset $\mathcal{U}$ of the space of valid model parameters, the Implicit Function Theorem implies that $\mathcal{E}^p(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r})$ and $\mathcal{E}_i(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r})$ are real-analytic functions. Assume by contradiction that $\mathcal{M}$ has a non-empty interior. This implies the subset $\mathcal{M}_i^p(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r}) := \alpha_i^p + g\mathcal{E}^p(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r}) - r_i\mathcal{E}_i(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r})$ vanishes on an open set within $\mathcal{U}$ for some (i, p). By the Identity Theorem for real-analytic functions, $\mathcal{M}_i^p(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r}) \equiv 0$ everywhere on $\mathcal{U}$.

By continuity, there exists some $\epsilon > 0$ sufficiently small such that $r_j = \epsilon$ for all workers implies every worker is fully active and fully utilized at every project. With vanishing cross-project costs, each worker treats each project independently. Since V is strictly increasing in each argument, the principal assigns each worker to each project and pushes them to exert the highest feasible effort, made possible by fully utilizing them. Thus, $\mathcal{M}_i^p(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r}) > 0$ for a set of model parameters

$(\boldsymbol{\alpha}, \mathbf{c}, \mathbf{g}, \mathbf{r})$ with positive measure. Thus $\mathcal{M}_i^p$ cannot contain an open set and must be measure zero. Thereby, $\mathcal{M} = \cup_{i,p}\mathcal{M}_i^p$ also has measure zero. We conclude that $e_i^p = 0$ implies $\lambda_i^p = 0$ is a generic property of the solution and can be assumed without loss of generality.